Structural Optimization **17**, 86–94 © Springer-Verlag 1999

Research Papers

On optimal microstructures for a plane shape optimization problem

G. Allaire and S. Aubry

Laboratoire d'Analyse Numérique, Université Paris 6, 4, place Jussieu, F-75252 Paris Cedex 05, France

Abstract The purpose of this article is to discuss some properties of optimal microstructure that are used in the homogenization approach for structural optimization problems involving compliance as the design criterion. The key ingredient for the homogenization method is to allow for microperforated composite materials as admissible designs. Some authors use periodic holes while others rely on optimal microstructures such as the so-called rank-2 layered materials which achieve optimality in the 2-D Hashin-Shtrikman bound. We prove that, in two space dimension, when the eigenvalues of the average stress have opposite signs, there is no optimal periodic microstructure. We also prove in this case that any optimal microstructure is degenerate, like the rank-2 layered material, i.e. it cannot sustain a nonaligned shear stress. When the eigenvalues of the average stress have the same sign, we exhibit higher order layered material that is optimal and not degenerate.

1 Introduction

Solving structural optimization problems by the homogenization method, amounts to find extremal microstructures which maximize the rigidity of a structure or equivalently which minimize its compliance (the work done by the load the structure is submitted to). These microstructures are called extremal in the sense that they achieve optimality in the well-known Hashin-Shtrikman bounds on the effective properties of composite materials. For more details on the homogenization method in structural design, we refer to Allaire *et al.* (1997), Allaire and Kohn (1993b), Bendsøe (1995), Bendsøe and Kikuchi (1988), Gibianski and Cherkaev (1997), Jog *et al.* (1994), Kohn and Strang (1986), Murat and Tartar (1997), and references therein. There are several examples of optimal microstructures in the literature. Mainly, they are the sequential laminates (see e.g. Francfort and Murat 1986), the concentric sphere assemblages of Hashin (1963), the confocal ellipsoid assemblages of Tartar [Tartar (1985), and Grabovsky and Kohn (1995a) in the elasticity setting], the Vigdergauz periodic constructions (Vigdergauz 1994; Grabovsky and Kohn 1995b).

Before discussing the properties of these extremal microstructures, we introduce the shape optimization problem considered in this paper. We restrict ourselves to plane problems, corresponding to the generalized shape optimization problem for perforated plates in plane stress [according to the terminology of Rozvany *et al.* (1995)]. We seek the optimal shape of a linearly elastic structure that minimizes a weighted sum of its compliance and weight. More precisely, let Ω be a bounded domain of $\mathbb{R}^2$ submitted to a surfacic load $\mathbf{f}$ on its boundary and occupied by an isotropic elastic material, characterized by a bulk modulus κ and a shear modulus μ. An admissible structure ω is a subset of the reference domain Ω obtained by removing some holes. The equations of elasticity for the resulting design are

$$\begin{cases} \boldsymbol{\sigma}_\omega = \mathbf{A}\mathbf{e}(\mathbf{u}_\omega), \qquad \mathbf{e}(\mathbf{u}_\omega) = \frac{1}{2}(\nabla \mathbf{u}_\omega + \nabla \mathbf{u}_\omega^T)\,, \\ \operatorname{div} \boldsymbol{\sigma}_\omega = 0 \text{ in } \omega\,, \\ \boldsymbol{\sigma}_\omega . n = \mathbf{f} \text{ on } \partial\Omega\,, \\ \boldsymbol{\sigma}_\omega . \mathbf{n} = 0 \text{ on } \partial\omega \setminus \partial\Omega\,, \end{cases} \tag{1}$$

where $\mathbf{u}_\omega$ is the displacement vector, $\mathbf{e}(\mathbf{u}_\omega)$ is the strain tensor, and $\boldsymbol{\sigma}_\omega$ is the stress tensor. The compliance of the structure is defined by

$$c(\omega) = \int_{\partial\Omega} \mathbf{f}.\mathbf{u}_\omega = \int_\omega \mathbf{A}\mathbf{e}(\mathbf{u}_\omega).\mathbf{e}(\mathbf{u}_\omega) = \int_\omega \mathbf{A}^{-1}\boldsymbol{\sigma}_\omega \cdot \boldsymbol{\sigma}_\omega\,.$$

Our structural optimization problem is to minimize, over all subsets $\omega \subset \Omega$, the objective function $\mathrm{E}(\omega)$ equal to the weighted sum of the compliance and weight of ω

$$\mathrm{E}(\omega) = c(\omega) + \lambda|\omega|\,.$$

It can be written as

$$\min_{\omega\subset\Omega} \mathrm{E}(\omega)\,.$$

Since this problem is known to generically have no solution [cf. the counter-examples of Murat (1977) and the numerical evidence of Cheng and Olhoff (1981)], we shall work with its relaxation. This means that we must enlarge the space of admissible designs by permitting perforated composites as solutions. Such composite structures are determined by two functions $\theta(\mathbf{x})$ and $\mathbf{A}^*(\mathbf{x})$: θ is the local volume fraction of the original material, taking values between 0 and 1, and $\mathbf{A}^*$ is the effective Hooke's law determined by the microstructure of perforations.

The relaxation formulation of (1) is given by

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{A}^*\mathbf{e}(\mathbf{u}) \text{ in } \Omega\,, \\ \operatorname{div} \boldsymbol{\sigma} = 0 \text{ in } \Omega\,, \\ \boldsymbol{\sigma}.\mathbf{n} = \mathbf{f} \text{ on } \partial\Omega\,. \end{cases} \tag{2}$$

The compliance is now

$$\bar{c}(\theta, \mathbf{A}^*) = \int_{\partial\Omega} \mathbf{f}.\mathbf{u} = \int_{\Omega} \mathbf{A}^{*-1}\boldsymbol{\sigma} \cdot \boldsymbol{\sigma},$$

and the objective function rewrites

$$\overline{E}(\theta, \mathbf{A}^*) = \bar{c}(\theta, \mathbf{A}^*) + \lambda \int_{\Omega} \theta .$$

For a given value θ of the density, they are many different effective Hooke's law $\mathbf{A}^*$ in a set G_θ, the so-called G-closure set at volume fraction θ, which is the set of all possible homogenized Hooke's law with density θ. The relaxed optimization problem is thus

$$\min_{0\leq\theta\leq 1,\ \mathbf{A}^*\in G_\theta} \overline{E}(\theta, \mathbf{A}^*) .$$

The main advantages of this relaxed problem are the existence of minimizers and the efficient numerical algorithms available for computing them. For more details on the theoretical aspects of this relaxation procedure we refer to Allaire *et al.* (1997), Allaire and Kohn (1993b), Bendsøe (1995), Gibianski and Cherkaev (1997), Kohn and Strang (1986) and Murat and Tartar (1997), and for the numerical aspects to Allaire *et al.* (1997), Bendsøe and Kikuchi (1988), and Jog *et al.* (1994).

Although the set G_θ is unknown, it is possible to compute the minimum over G_θ of the complementary energy $\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$. The minimum value is called a Hashin-Shtrikman bound, and the homogenized Hooke's laws $\mathbf{A}^*$ that achieve this minimum correspond to so-called optimal microstructures. Such a computation is recalled in Section 2. An optimal microstructure can always be found in the class of rank-2 sequential laminates aligned with the principal directions of the stress $\boldsymbol{\sigma}$ (its eigenvectors). However, rank-2 sequential laminates have the inconvenience of having a degenerate Hooke's law: in the basis of the principal stress directions, its rigidiy tensor has a zero component, $\mathbf{A}^*_{1212} = 0$. In other words, this rank-2 sequential laminate cannot sustain a shear stress not aligned with $\boldsymbol{\sigma}$. Although this does not mean that this structure is unstable for any load condition, it is generically impossible to solve the corresponding elasticity equations. This fact yields some difficulties in the numerical algorithm used by Allaire *et al.* (1997) (see also Bendsøe 1995), since it is based on iteratively solving elasticity problems for the previous optimal microstructures. Such a difficulty is peculiar to the 2-D case because the optimal rank-3 sequential laminates in 3-D do not suffer from this degeneracy property.

The aim of this note is to prove that, in two-dimensional space, when the average stress $\boldsymbol{\sigma}$ is such that $\det\boldsymbol{\sigma} \leq 0$, any optimal microstructure is degenerate. On the other hand, when $\det\boldsymbol{\sigma} > 0$, we exhibit a higher order sequential laminate which is both optimal and nondegenerate. In other words, the numerical problem of dealing with nondefinite Hooke's law cannot be alleviated if $\det\boldsymbol{\sigma} \leq 0$, while a simple remedy is available if $\det\boldsymbol{\sigma} > 0$. Other extremal microstructures have been investigated by Grabovsky and Kohn (1995a,b), namely the concentric spheres, or confocal ellipsoids, assemblages and Vigdergauz periodic constructions. Note however that they are optimal only in the case $\det\boldsymbol{\sigma} > 0$ (and their Hooke's law is not explicit). Our method relies on a careful investigation of the Hashin-Shtrikman bound on the effective elastic energy. As such, its application is restricted to the compliance optimization problem. We are therefore unable to extend our results to so-called nonselfadjoint problems for which the objective function is not the compliance.

With this goal in view, we first prove that, if $\det\boldsymbol{\sigma} < 0$, there is no periodic microstructure which is optimal [this is consistent with recent results by Cherkaev *et al.* (1998)]. Then, in the same case, we show that the microgeometry of a nonperiodic optimal microstructure must have an H-measure with the same support as that of the optimal rank-2 sequential laminate (the H-measure is a kind of two-point correlation function, see the proof of Theorem 2 for a precise definition). This implies that any optimal composite cannot sustain a nonaligned shear stress when $\det\boldsymbol{\sigma} < 0$. Finally, we show that, when $\det\boldsymbol{\sigma} > 0$, there exists a rank-4 laminate that is optimal and nondegenerate.

2 Hashin-Shtrikman lower bound for complementary energy

In this section we recall classical results about Hashin-Shtrikman bounds on the effective energy of a two-phase composite material. In the next sections we shall use several technical points introduced here. We follow the exposition of Allaire and Kohn (1993a,b), and Kohn (1991) (see also the references therein).

Let us introduce some notations before proceeding. Let S be the space of all 2×2 symmetric second-order tensors. The inner products in $\mathbb{R}^2$ and in S are represented by a dot. We denote by $\otimes$ the tensor product and by $\odot$ the symmetrized tensor product defined by

$$\mathbf{a}\odot\mathbf{b} = \frac{1}{2}(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})\ \forall \mathbf{a},\mathbf{b}\in\mathbb{R}^2 .$$

Let G_θ be the set of all effective Hooke's law corresponding to a microperforated composite obtained by mixing the original material $\mathbf{A}$ with void in proportions θ and $1-\theta$, respectively. Since homogenization is a local process, i.e. at each point $\mathbf{x}$ the value of $\theta(\mathbf{x})$ and $\mathbf{A}^*(\mathbf{x})$ do not depend on what happens elsewhere, we restrict our attention to constant values of the density and of the homogenized Hooke's law (of course, in the shape optimization process, these values may vary from point to point). In truth, homogenization can be made rigorous only for mixtures of two nondegenerate materials. Therefore, we fill the holes and replace void by a very compliant material $\mathbf{B}$. We prove all our results for such two-phase composites, and in the end let $\mathbf{B}$ go to 0 to mimick holes. We shall not justify further this approach and refer to Allaire *et al.* (1997) for a thorough discussion of this matter. We first recall, without proof, a classical result of the homogenization theory (due to Dal Maso and Kohn) which claims that there is no loss of generality in considering only composites obtained by homogenization of periodic microstructures.

Proposition 1 *Let P_θ be the set of all periodic composites defined, for $0 \le \theta \le 1$, by $\mathbf{A}^* \in P_\theta$ if and only if there exists a periodic characteristic function $\chi(\mathbf{y})$ such that*

$$\int_Y \chi(\mathbf{y})\,d\mathbf{y} = \theta\,, \tag{3}$$

and, for any stress $\boldsymbol{\sigma} \in S$, $\mathbf{A}^*$ is defined by the following quadratic form:

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \inf_{\substack{\mathrm{div}\eta=0\\ \int_Y \eta(\mathbf{y})\,d\mathbf{y}=0}} \int_Y \Big[(1-\chi(\mathbf{y}))\mathbf{B}^{-1} + \chi(\mathbf{y})\mathbf{A}^{-1}\Big](\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y}))\cdot(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y}))\,d\mathbf{y}\,. \tag{4}$$

Then, G_θ is the closure of P_θ in the space of fourth-order symmetric tensors.

In the limit when $\mathbf{B}$ goes to 0, definition (4) of the homogenized tensor $\mathbf{A}^*$ implies that the stress $\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y})$ must vanish in the holes. Before giving the Hashin-Shtrikman bound, we introduce the class of sequential laminate composites with their explicit formula (5) (see Francfort and Murat 1986).

Definition 1 *A rank-p sequential laminate of material $\mathbf{A}$ and void in proportions θ and $1-\theta$, respectively, with unit lamination directions $(e_i)_{1\le i\le p}$ and lamination parameters $(m_i)_{1\le i\le p}$ satisfying $0 \le m_i \le 1$ and $\sum_{i=1}^p m_i = 1$, is defined by its Hooke's law $\mathbf{A}_L$*

$$(1-\theta)(\mathbf{A}_L^{-1} - \mathbf{A}^{-1})^{-1} = \theta \sum_{i=1}^{p} m_i f_{\mathbf{A}}(e_i)\,, \tag{5}$$

where $f_{\mathbf{A}}(e)$ is a positive nondefinite fourth-order tensor defined, for any symmetric tensor $\boldsymbol{\xi}$, by

$$f_{\mathbf{A}}(e)\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi} - 4\mu\,[(\boldsymbol{\xi}e)\odot e - (\boldsymbol{\xi}e.e)e\odot e] - \frac{1}{\kappa+\mu}\,[(\kappa-\mu)tr(\boldsymbol{\xi}) + 2\mu(\boldsymbol{\xi}e.e)]\,[(\kappa+\mu)\mathbf{I}_2 + 2\boldsymbol{\mu}e\odot e]\,. \tag{6}$$

The set of all sequential laminates of rank p, denoted by L^p_θ, is a subset of G_θ.

We obtain an optimal bound for the complementary energy evaluated at the stress $\boldsymbol{\sigma}$ by using the well-known Hashin-Shtrikman variational principle.

Proposition 2 *For $0 \le \theta \le 1$, any $\mathbf{A}^*$ in G_θ satisfies*

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \ge HS(\boldsymbol{\sigma})\,,$$

where $HS(\boldsymbol{\sigma})$ is the so-called Hashin-Shtrikman bound defined by

$$HS(\boldsymbol{\sigma}) = \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + (1-\theta)\max_{\boldsymbol{\tau}\in S}\{2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \theta g(\boldsymbol{\tau})\}\,, \tag{7}$$

with $g(\boldsymbol{\tau}) = \max_{|\mathbf{k}|=1} G(\boldsymbol{\tau},\mathbf{k})$ and

$$G(\boldsymbol{\tau},\mathbf{k}) = \frac{4\kappa\mu}{\kappa+\mu}(|\boldsymbol{\tau}|^2 - 2|\boldsymbol{\tau}\mathbf{k}|^2 + (\boldsymbol{\tau}\mathbf{k}.\mathbf{k})^2)\,.$$

Furthermore, denoting by σ_1, σ_2 the eigenvalues of $\boldsymbol{\sigma}$, and by $\mathbf{n}_1, \mathbf{n}_2$ the corresponding unit eigenvectors, the maximum in (7) is achieved by

$$\boldsymbol{\tau} = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\sigma_1| + |\sigma_2|)(\mathbf{n}_1\otimes\mathbf{n}_1 - \mathbf{n}_2\otimes\mathbf{n}_2) \text{ if } det\boldsymbol{\sigma} \le 0\,,$$

$$\boldsymbol{\tau} = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\sigma_1| + |\sigma_2|)\mathbf{I}_2 \text{ if } det\boldsymbol{\sigma} > 0\,,$$

and, this bound is attained by a rank-2 laminate.

Proof: Since the set P_θ of periodic composites is dense in the set G_θ of all possible composites, we establish the lower bound for $\mathbf{A}^* \in P_\theta$. We first fill the holes with a weak material $\mathbf{B}$ that will tend to 0 in the end. Denoting by $Y = [0,1]^2$ the periodic cell, and using a dual variational formulation (i.e. complementary energy), the Hooke's law $\mathbf{A}^*$ of a periodic composite is defined by the following formula:

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \inf_{\substack{\mathrm{div}\eta=0\\ \int_Y \eta(\mathbf{y})\,d\mathbf{y}=0}} \int_Y \Big[(1-\chi(\mathbf{y}))\mathbf{B}^{-1} + \chi(\mathbf{y})\mathbf{A}^{-1}\Big](\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y})).(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y}))\,d\mathbf{y}\,,$$

where $\chi(\mathbf{y})$ is the characteristic function of that part of Y occupied by material $\mathbf{A}$. The volume fraction of material $\mathbf{A}$ is then

$$\theta = \int_Y \chi(\mathbf{y})\,d\mathbf{y}\,.$$

Step 1. We begin by adding and subtracting a reference energy

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \inf_{\substack{\mathrm{div}\eta=0\\ \int_Y \eta(\mathbf{y})\,d\mathbf{y}=0}} \Bigg[\int_Y (1-\chi(\mathbf{y}))(\mathbf{B}^{-1} - \mathbf{A}^{-1})(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y})).(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y})))\,d\mathbf{y} + \int_Y \mathbf{A}^{-1}(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y})).(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y}))\,d\mathbf{y}\Bigg]\,.$$

Using the positivity of $(\mathbf{B}^{-1} - \mathbf{A}^{-1})$ and convex duality, the first integral on the right-hand side is rewritten

$$\sup_{\boldsymbol{\tau}(\mathbf{y})} \int_Y (1-\chi(\mathbf{y}))(2(\boldsymbol{\sigma}+\boldsymbol{\eta}(\mathbf{y}))\cdot\boldsymbol{\tau}(\mathbf{y}) - (\mathbf{B}^{-1} - \mathbf{A}^{-1})^{-1}\boldsymbol{\tau}(\mathbf{y})\cdot\boldsymbol{\tau}(\mathbf{y}))\,d\mathbf{y} \tag{8}$$

Step 2. A lower bound is obtained in (8) by setting $\boldsymbol{\tau}(\mathbf{y}) = \boldsymbol{\tau}$ independent of $\mathbf{y}$. This yields

$$2(1-\theta)\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - (1-\theta)(\mathbf{B}^{-1} - \mathbf{A}^{-1})^{-1}\boldsymbol{\tau}\cdot\boldsymbol{\tau} + \int_Y 2(1- \chi(\mathbf{y}))\boldsymbol{\eta}(\mathbf{y})\cdot\boldsymbol{\tau}\,d\mathbf{y}\,, \tag{9}$$

where $\theta = \int_Y \chi(\mathbf{y})\,d\mathbf{y}$. After some algebra, we have

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - (1-\theta)(\mathbf{B}^{-1} - \mathbf{A}^{-1})^{-1}\boldsymbol{\tau}\cdot\boldsymbol{\tau} + 2(1-\theta)\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \min_{\substack{\mathrm{div}\eta=0\\ \int_Y \eta(\mathbf{y})\,d\mathbf{y}=0}} \int_Y \Big[-2\,\chi(\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\cdot\boldsymbol{\tau} + \mathbf{A}^{-1}\boldsymbol{\eta}(\mathbf{y})\cdot\boldsymbol{\eta}(\mathbf{y})\,d\mathbf{y}\Big]$$

By taking the limit $\mathbf{B}\to 0$, we have

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + 2(1-\theta)\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \min_{\substack{\mathrm{div}\eta=0\\ \int_Y \eta(\mathbf{y})\,d\mathbf{y}=0}} \int_Y \Big[-2\,\chi(\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\cdot\boldsymbol{\tau} + \mathbf{A}^{-1}\boldsymbol{\eta}(\mathbf{y})\cdot\boldsymbol{\eta}(\mathbf{y})\,d\mathbf{y}\Big]\,.$$

Step 3. The last minimum with respect to $\boldsymbol{\eta}$ is called the nonlocal term. It can be computed by Fourier analysis. Denoting by $\widehat{\chi}(\mathbf{k})$ the Fourier component at the frequency $\mathbf{k}$ of the characteristic function χ, i.e.

$$\chi(\mathbf{y}) = \sum_{\mathbf{k}\in Z^N} \widehat{\chi}(\mathbf{k})e^{2i\pi\mathbf{k}.\mathbf{y}}\,,$$

this last integral is exactly equal to

$$-\sum_{\mathbf{k}\neq 0} |\widehat{\chi}(\mathbf{k})|^2 G(\boldsymbol{\tau},\mathbf{k})\,, \tag{10}$$

where

$$G(\boldsymbol{\tau},\mathbf{k}) = \frac{4\kappa\mu}{\kappa+\mu}\left(|\boldsymbol{\tau}|^2 - 2\frac{|\boldsymbol{\tau}\mathbf{k}|^2}{|\mathbf{k}|^2} + \frac{(\boldsymbol{\tau}\mathbf{k}.\mathbf{k})^2}{|\mathbf{k}|^4}\right).$$

Remarking that

$$\sum_{\mathbf{k}\neq 0} |\widehat{\chi}(\mathbf{k})|^2 = \theta(1-\theta)\,, \tag{11}$$

the quantity (10) is bounded from below by

$$-\sum_{\mathbf{k}\neq 0} |\widehat{\chi}(\mathbf{k})|^2 G(\boldsymbol{\tau},\mathbf{k}) \geq -\theta(1-\theta)g(\boldsymbol{\tau})\,, \tag{12}$$

where $g(\boldsymbol{\tau}) = \max_{|\mathbf{k}|=1} G(\boldsymbol{\tau},\mathbf{k})$. Varying $\boldsymbol{\tau}$ among all constant symmetric matrices gives the desired lower bound

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + (1-\theta)\sup_{\boldsymbol{\tau}\in S}\{2\boldsymbol{\tau}\cdot\boldsymbol{\sigma} - \theta g(\boldsymbol{\tau})\}\,.$$

To evaluate $g(\boldsymbol{\tau})$, we maximize $G(\boldsymbol{\tau},\mathbf{k})$ on the unit sphere. Decomposing the vector $\mathbf{k}$ on the eigenbasis of the symmetric matrix $\boldsymbol{\tau}$, an easy computation yields

$$g(\boldsymbol{\tau}) = \frac{4\kappa\mu}{\kappa+\mu}\max(\tau_1^2,\tau_2^2)\,,$$

where τ_1 and τ_2 are the two eigenvalues of $\boldsymbol{\tau}$. Of course, the corresponding optimal $\mathbf{k}$'s are eigenvectors of $\boldsymbol{\tau}$ associated with the eigenvalue of largest absolute value.

Step 4. To prove the attainability of the bound (7), for each stress $\boldsymbol{\sigma}$, we exhibit a rank-2 sequential laminate with Hooke's law $\mathbf{A}_L$ such that $\mathbf{A}_L^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = HS(\boldsymbol{\sigma})$. As is well-known, the optimality condition with respect to $\boldsymbol{\tau}$ in the definition of $HS(\boldsymbol{\sigma})$ delivers the lamination parameters of the optimal sequential laminate. Since $g(\boldsymbol{\tau})$ depends only on the eigenvalues of $\boldsymbol{\tau}$, the optimal $\boldsymbol{\tau}$ in

$$\max_{\boldsymbol{\tau}\in S}\{2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \theta g(\boldsymbol{\tau})\} \tag{13}$$

must be simultaneously diagonal with $\boldsymbol{\sigma}$ because this is so for the inner product $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$. For $\sigma_1,\sigma_2\neq 0$, the optimal choice is easily seen to be $\tau_1 = \mathrm{sgn}\,(\sigma_1)t$ and $\tau_2 = \mathrm{sgn}\,(\sigma_2)t$ with

$$t = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\sigma_1| + |\sigma_2|)\,. \tag{14}$$

Recall that the fourth-order tensor $f_{\mathbf{A}}(\mathrm{e})$ defined by (6) is such that

$$f_{\mathbf{A}}(\mathrm{e})\boldsymbol{\tau}\cdot\boldsymbol{\tau} = G(\boldsymbol{\tau},\mathrm{e})\,.$$

Therefore, the optimal $\boldsymbol{\tau}$ is easily seen to satisfy

$$\boldsymbol{\sigma} = \theta\sum_{i=1}^{2} m_i f_{\mathbf{A}}(\mathbf{n}_i)\boldsymbol{\tau}\,, \tag{15}$$

where $\mathbf{n}_1,\mathbf{n}_2$ are the unit eigenvectors of $\boldsymbol{\sigma}$, associated to the eigenvalues σ_1,σ_2, and

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}\,, \qquad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}\,. \tag{16}$$

In other words, recalling (5), there exists a rank-2 sequential laminate A_L such that

$$\sigma = (1-\theta)\left(\mathbf{A}_L^{-1} - \mathbf{A}^{-1}\right)^{-1} \tau .$$

Since, for the optimal τ, we have

$$HS(\sigma) = \mathbf{A}^{-1}\sigma \cdot \sigma + (1-\theta)\sigma \cdot \tau ,$$

this proves that $HS(\sigma) = \mathbf{A}_L^{-1}\sigma \cdot \sigma$. When one of the eigenvalues of σ is zero, say $\sigma_2 = 0$, an optimal τ is only constrained by $\tau_1 = \frac{\kappa+\mu}{4\kappa\mu\theta}\sigma_1$ and $|\tau_2| \leq |\tau_1|$. In this case, m_1 is zero and the optimal rank-2 sequential laminate is truly a rank-1 sequential laminate.

Remark 1 *When* $\det\sigma < 0$, *the optimal* τ *in (13) is* $\tau = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, *and it has two distinct eigenvalues. In this case,* $\mathbf{k}$ *maximizes* $G(\tau, \mathbf{k})$ *if and only if it is one of the two eigenvectors of* σ. *As we shall see in the next section, this fact implies more or less that there is no other optimal microstructure apart from the rank-2 sequential laminate. When* $\det\sigma > 0$, *the optimal* τ *in (13) is* $\tau = t\mathbf{I}_2$. *Then,* $G(\tau, \mathbf{k})$ *is constant and every* $\mathbf{k}$ *is a maximizer. In the next section, we will check that this freedom of choice for the vector* $\mathbf{k}$ *allows several other microstructures to be optimal. Finally, when* $\det\sigma = 0$ *(and* $\sigma \neq 0$*), the optimal* τ *is not unique but it can be chosen to be proportional to* σ. *Then,* k *maximizes* $G(\tau, \mathbf{k})$ *if and only if it is an eigenvector of* σ *for its unique nonzero eigenvalue.*

3 Main results

We can now state the new results of this article.

Theorem 1 *Let* $\mathbf{A}^*$ *be the Hooke's law in* G_θ *of an optimal microstructure for the Hashin-Shtrikman lower bound (7), i.e.*

$$\mathbf{A}^{*-1}\sigma \cdot \sigma = HS(\sigma) .$$

If $\det\sigma < 0$, *then* $\mathbf{A}^* \notin P_\theta$, *i.e. an optimal microstructure cannot be obtained by periodic homogenization. If* $\det\sigma = 0$, *then* $\mathbf{A}^* \in P_\theta$ *if and only if it is a rank-1 sequential laminate.*

Remark 2 *As a rank-2 laminate is not a periodic microstructure, the previous theorem does not contradict Proposition 2 which claims that the Hashin-Shtrikman bound is always achieved by a rank-2 laminate. Indeed, such a structure is defined as two successive simple laminations occurring at well separated lengthscales. It is therefore not periodic since periodic microstructures have a single lengthscale.*

The fact that the only optimal composite, when σ *is uniaxial in 2-D (which is equivalent to* $\det\sigma = 0$*), is a rank-1 sequential laminate is already well-known (see Ball and James 1987).*

Theorem 2 *Let* $\mathbf{A}^*$ *be the Hooke's law in* G_θ *of an optimal microstructure for the Hashin-Shtrikman lower bound (7), i.e.*

$$\mathbf{A}^{*-1}\sigma \cdot \sigma = HS(\sigma) .$$

If $\det\sigma \leq 0$, *then,* $\mathbf{A}^*$ *is degenerate, as the rank-2 sequential laminate, in the sense that, in the eigenbasis of* σ, *it satisfies*

$$\left(\mathbf{A}^{*-1}\right)_{1212} = +\infty ,$$

i.e. it cannot sustain a nonaligned shear stress.

Remark 3 *Actually, the proof of Theorem 2 amounts to show that the* H*-measure of the optimal* $\mathbf{A}^*$ *[see Tartar (1990) and (19) for a precise definition] has the same support as that of the optimal rank-2 sequential laminate when* $\det\sigma \leq 0$. *In particular it implies that, if* $\det\sigma < 0$, *there is no other optimal sequential laminate apart from the rank-2 one exhibited in the previous section, and if* $\det\sigma = 0$, *the only optimal microstructure is a rank-1 laminate. On the contrary, if* $\det\sigma > 0$, *the next result states that there are other optimal sequential laminates which can even be nondegenerate.*

Remark 4 *In a recent paper, Cherkaev et al. (1998) found the optimal shape of a simply connected hole in an infinite elastic plane submitted to a shear stress at infinity (i.e.* $\det\sigma < 0$*). This problem is equivalent, in the low volume limit (for the hole), to that of finding the optimal periodic microstructure under the restriction that the hole is simply connected (which is not the case for the rank-two laminate). Of course, the homogenized properties of such a perforated microstructure is not degenerate. However, they remarked [see Section 1.2 in the paper by Cherkaev et al. (1998)] that the energy of their microstructure is significantly higher than that of the rank-two laminate, which is consistent with our Theorem 2.*

Proposition 3 *Assume that* $\det\sigma > 0$. *Denoting by* σ_1, σ_2 *the eigenvalues of* σ, *and by* n_1, n_2 *the corresponding unit eigenvectors, there exists a nondegenerate rank-4 laminate, achieving optimality in the Hashin-Shtrikman bound (7), which is defined by the lamination directions*

$$\mathbf{e}_1 = \mathbf{n}_1, \quad \mathbf{e}_2 = \mathbf{n}_2, \quad \mathbf{e}_3 = \frac{\mathbf{n}_1 + \mathbf{n}_2}{|\mathbf{n}_1 + \mathbf{n}_2|}, \quad \mathbf{e}_4 = \frac{\mathbf{n}_1 - \mathbf{n}_2}{|\mathbf{n}_1 - \mathbf{n}_2|},$$

and the lamination parameters

$$m_1 = \frac{|\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_2 = \frac{2|\sigma_1| - |\sigma_2|}{2(|\sigma_1| + |\sigma_2|)},$$
$$m_3 = m_4 = m_1 ,$$

where, without loss of generality, we have assumed that $|\sigma_1| \geq |\sigma_2|$.

Remark 5 *The condition* $\det\sigma > 0$ *for having optimal higher rank sequential laminate is not a surprise. This is precisely the assumption used by Grabovsky and Kohn (1995a,b) to show that the so-called confocal ellipsoid assemblages of Tartar (1995) and the periodic constructions of Vigdergauz (1994) are two other types of optimal microstructures for the Hashin-Shtrikman bound. Of course, such composites are not optimal when* $\det\sigma < 0$, *and when* $\det\sigma = 0$ *they all degenerate to the optimal rank-1 sequential laminate.*

Proof of Theorem 1 Let $\mathbf{A}^*$ be a periodic microstructure which achieves optimality in the Hashin-Shtrikman bound, i.e.

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = HS(\boldsymbol{\sigma}).$$

During the proof of Proposition 2, the Hashin-Shtrikman bound has been established by making two inequalities in (9) (Step 2) and (11) (Step 3). Actually, for an optimal tensor $\mathbf{A}^*$ these inequalities *must be equalities*. We focus on (11) which becomes the following equality:

$$\sum_{\mathbf{k}\neq 0} |\widehat{\chi}(\mathbf{k})|^2 G(\boldsymbol{\tau},\mathbf{k}) = \theta(1-\theta)\max_{|\mathbf{k}|=1} G(\boldsymbol{\tau},\mathbf{k}), \tag{17}$$

where, for $|\mathbf{k}|=1$,

$$G(\boldsymbol{\tau},\mathbf{k}) = \frac{4\kappa\mu}{\kappa+\mu}\left[|\boldsymbol{\tau}|^2 - 2|\boldsymbol{\tau}\mathbf{k}|^2 + (\boldsymbol{\tau}\mathbf{k}.\mathbf{k})^2\right].$$

When $\det\boldsymbol{\sigma}<0$ the unique optimal $\boldsymbol{\tau}$ in (7) (see Proposition 2) is equal to

$$\boldsymbol{\tau} = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\sigma_1|+|\sigma_2|)(\mathbf{n}_1\otimes\mathbf{n}_1 - \mathbf{n}_2\otimes\mathbf{n}_2).$$

As $\boldsymbol{\tau}$ is not proportional to the identity matrix, $G(\boldsymbol{\tau},\mathbf{k})$ is not independent of $\mathbf{k}$ and, for any $\mathbf{k}$ such that $G(\boldsymbol{\tau},\mathbf{k}) < g(\boldsymbol{\tau})$, the equality (17) shows that necessarily

$$\widehat{\chi}(\mathbf{k}) = 0.$$

Because for all directions $\mathbf{k}$ that are not eigenvectors of $\boldsymbol{\sigma}$ (there are only two such eigenvectors $\mathbf{n}_1$ and $\mathbf{n}_2$) we have $G(\boldsymbol{\tau},\mathbf{k}) < g(\boldsymbol{\tau})$, we obtain a contradiction: for example, taking $\mathbf{k}$ parallel to $\mathbf{n}_1+\mathbf{n}_2$, implies that $\chi(\mathbf{y})$ does not depend on $\mathbf{y}\cdot(\mathbf{n}_1+\mathbf{n}_2)$, i.e. it is invariant in the $\mathbf{n}_1+\mathbf{n}_2$ direction. This would imply that χ is the characteristic function of a single lamination of $\mathbf{A}$, which is impossible because it is clearly not optimal.

When $\det\boldsymbol{\sigma}=0$ (say $\sigma_2=0$), the optimal $\boldsymbol{\tau}$ is not unique but it can be chosen equal to $t\mathbf{n}_1\otimes\mathbf{n}_1$. From (17) we deduce again that $\chi(\mathbf{y})$ depends only on $\mathbf{y}\cdot\mathbf{n}_1$ because $\boldsymbol{\tau}$ is a rank-1 matrix. This implies that $\chi(\mathbf{y})$ is the characteristic function of a rank-1 laminate. This proves the theorem.

Proof of Theorem 2 Let $\mathbf{A}^*$ be a nonperiodic microstructure which achieves optimality in the Hashin-Shtrikman bound, i.e.

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = HS(\boldsymbol{\sigma}).$$

Since the set P_θ of periodic composites is dense in G_θ (see Proposition 1), there exists a sequence $\mathbf{A}_n^*\in P_\theta$ such that $\mathbf{A}_n^*$ converges to $\mathbf{A}^*$, in the space of fourth-order tensors. Let $\chi_n(\mathbf{y})$ be the characteristic function associated to $\mathbf{A}_n^*\in P_\theta$. It satisfies

$$\int_\Omega \chi_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = \theta. \tag{18}$$

For each characteristic function χ_n, or equivalently for each composite $\mathbf{A}_n^*$, we introduce a probability measure μ_n defined on the unit sphere $S_1=\{k\in I\!R^2,\ |k|=1\}$ with values in $I\!R$, that is called the H-measure (see the work of Tartar 1990), given by

$$\mu_n(\mathrm{e}) = \frac{1}{\theta(1-\theta)}\sum_{\mathbf{k}\in Z,\,\mathbf{k}\neq 0} |\widehat{\chi}_n(\mathbf{k})|^2\delta\left(\frac{\mathbf{k}}{|\mathbf{k}|}-\mathrm{e}\right), \tag{19}$$

where δ is the Dirac mass at the origin. As defined by (19), μ_n is a probability measure because

$$\mu_n(\mathrm{e})\geq 0 \quad\text{and}\quad \int_{S_1}\mu_n(\mathrm{e})\,\mathrm{d}\mathrm{e} = \frac{1}{\theta(1-\theta)}\sum_{\mathbf{k}\neq 0}|\widehat{\chi}_n(\mathbf{k})|^2 = 1.$$

The H-measure μ_n can be physically interpreted as a kind of two-points correlation function for the microstructure. It can be used to obtain a more precise bound than the Hashin-Shtrikman bound, as follows.

Lemma 1 *Let $\mathbf{A}_n^*$ be a periodic composite in P_θ. Let μ_n be its H-measure as defined by (19). Then, it satisfies*

$$\mathbf{A}_n^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} +$$
$$(1-\theta)\max_{\boldsymbol{\tau}\in S}\left[2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \theta\int_{|\mathbf{k}|=1} G(\boldsymbol{\tau},\mathbf{k})\mu_n(\mathbf{k})\,\mathrm{d}\mathbf{k}\right],$$

where $G(\boldsymbol{\tau},\mathbf{k})$ is defined by Proposition 2.

The proof of Lemma 1 is similar to that of Proposition 2 (on the Hashin-Shtrikman bound). The two first steps are identical, but in the third one, rather than optimizing over $\mathbf{k}$, we keep the exact expression of the nonlocal term. Then, by definition of the H-measure μ_n, we have

$$\sum_{\mathbf{k}\in Z,\mathbf{k}\neq 0} |\widehat{\chi}_n(\mathbf{k})|^2 G(\boldsymbol{\tau},\mathbf{k}) = \theta(1-\theta)\int_{|\mathbf{k}|=1} G(\boldsymbol{\tau},\mathbf{k})\mu_n(\mathbf{k})\,\mathrm{d}\mathbf{k},$$

which yields the desired result.

Coming back to the proof of Theorem 2, we apply Lemma 1, and we pass to the limit, when n goes to infinity. As is well-known, from any such sequence μ_n of probability measure (bounded and positive), one can extract a subsequence (still denoted by μ_n) and there exists another probability measure μ_∞ such that μ_n converges to μ_∞ weakly*. Therefore, for this subsequence the limit of Lemma 1 is

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \mathbf{A}^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} +$$
$$(1-\theta)\max_{\boldsymbol{\tau}\in S}\left[2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \theta\int_{S_1} G(\boldsymbol{\tau},\mathrm{e})\mu_\infty(\mathrm{e})\,\mathrm{d}\mathrm{e}\right], \tag{20}$$

where the left-hand side is precisely equal to the Hashin-Shtrikman bound $HS(\boldsymbol{\sigma})$. On the other hand, the function $g(\boldsymbol{\tau})$, introduced in Proposition 2, can also be defined in terms of H-measures by

$$g(\boldsymbol{\tau}) = \max_{\mu(\mathrm{e})\geq 0,\ \int_{S_1}\mu(\mathrm{e})\,\mathrm{d}\mathrm{e}=1}\int_{S_1} G(\boldsymbol{\tau},\mathrm{e})\mu(\mathrm{e})\,\mathrm{d}\mathrm{e}. \tag{21}$$

Therefore, we deduce a lower bound for (20)

$$HS(\sigma) = \mathbf{A}^{*-1}\sigma \cdot \sigma \geq \mathbf{A}^{-1}\sigma \cdot \sigma +$$

$$(1-\theta)\sup_{\tau}\left[2\sigma \cdot \tau - \theta g(\tau)\right] = HS(\sigma),$$

which is truly an equality. This implies that (20) is also an equality and μ_∞ is optimal in (21). According to the proof of Proposition 2, if $\det \sigma < 0$, τ is proportional to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and any optimal H-measure μ in (21) vanishes except when e is one of the two eigenvectors of τ and σ (which are simultaneously diagonal). Finally, we deduce that any limit H-measure μ_∞ has the same support identical to that of the H-measure of the optimal rank-2 sequential laminate.

It remains to check that, in such a case, the optimal Hooke's law $\mathbf{A}^*$ is also degenerate. In the eigenbasis of σ defining a nonaligned shear stress $\tilde{\sigma}$ by

$$\tilde{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we apply Lemma 1 at the new stress $\tilde{\sigma}$. We have

$$\mathbf{A}_n^{*-1}\tilde{\sigma} \cdot \tilde{\sigma} \geq \mathbf{A}^{-1}\tilde{\sigma} \cdot \tilde{\sigma} +$$

$$(1-\theta)\max_{\tilde{\tau} \in S}\left[2\tilde{\sigma} \cdot \tilde{\tau} - \theta \int_{|\mathbf{k}|=1} G(\tilde{\tau}, \mathbf{k})\mu_n(\mathbf{k})d\mathbf{k}\right],$$

and passing to the limit as n goes to infinity

$$\mathbf{A}^{*-1}\tilde{\sigma} \cdot \tilde{\sigma} \geq \mathbf{A}^{-1}\tilde{\sigma} \cdot \tilde{\sigma} +$$

$$(1-\theta)\max_{\tilde{\tau} \in S}\left[2\tilde{\sigma} \cdot \tilde{\tau} - \theta \int_{|\mathbf{k}|=1} G(\tilde{\tau}, \mathbf{k})\mu_\infty(\mathbf{k})d\mathbf{k}\right].$$

Writing $\tilde{\tau} = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$, a simple computation gives

$$\left(\mathbf{A}^{*-1}\right)_{1212} \geq \frac{1}{\mu} +$$

$$\max_{\tau_1,\tau_2,\tau_3}\left[4\theta\tau_3 - \frac{4\kappa\mu}{\kappa+\mu}\left(\tau_2^2\mu_\infty(\mathbf{n}_1) + \tau_1^2\mu_\infty(\mathbf{n}_2)\right)\right].$$

Clearly, fixing τ_1, τ_2 and letting τ_3 go to infinity shows that the right-hand side is unbounded, which implies that $\mathbf{A}^*$ is degenerate for this stress $\tilde{\sigma}$.

Proof of Proposition 3 We are seeking a rank-4 sequential laminate composite $\mathbf{A}_L^* \in L_\theta$ which saturates the Hashin-Shtrikman bound

$$\mathbf{A}_L^{*-1}\sigma \cdot \sigma = HS(\sigma) = \mathbf{A}^{-1}\sigma \cdot \sigma +$$

$$\frac{1-\theta}{\theta}\frac{\kappa+\mu}{4\kappa\mu}(|\sigma_1| + |\sigma_2|)^2 \tag{22}$$

By Definition 1, $\mathbf{A}_L^*$ satisfies

$$\mathbf{A}_L^{*-1}\sigma \cdot \sigma = \mathbf{A}^{-1}\sigma \cdot \sigma + \frac{1-\theta}{\theta}\left[\sum_{i=1}^{4} m_i f_{\mathbf{A}}(\mathrm{e}_i)\right]^{-1} \sigma \cdot \sigma, \tag{23}$$

with $f_{\mathbf{A}}(\mathrm{e}_i)$ defined by (6) and the lamination directions denoted by $(\mathrm{e}_i)_{i=1,4}$. Denoting by $(\mathbf{n}_1, \mathbf{n}_2)$ an orthonormal basis of eigenvectors of σ, we fix the unit lamination directions to be equal to

$$\mathrm{e}_1 = \mathbf{n}_1, \quad \mathrm{e}_2 = \mathbf{n}_2, \quad \mathrm{e}_3 = \frac{\mathbf{n}_1 + \mathbf{n}_2}{\sqrt{2}}, \quad \mathrm{e}_4 = \frac{\mathbf{n}_1 - \mathbf{n}_2}{\sqrt{2}}.$$

Our goal is to compute the parameters $(m_i)_{i=1,4}$ such that (22) holds true. The main task is to invert the (possibly degenerate) fourth-order tensor $\sum_{i=1}^{4} m_i f_{\mathbf{A}}(\mathrm{e}_i)$. For a unit vector $\mathbf{v}$, with components v_1, v_2 such that $v_1^2 + v_2^2 = 1$, and a symmetric matrix η, we have

$$f_{\mathbf{A}}(\mathbf{v})\eta =$$

$$\frac{4\kappa\mu}{\kappa+\mu}(v_2^2\eta_{11} - 2v_1v_2\eta_{12} + v_1^2\eta_{22})\begin{bmatrix} v_2^2 & -v_1v_2 \\ -v_1v_2 & v_1^2 \end{bmatrix}.$$

Applying it to each $\mathbf{e}_i$ and imposing $m_3 = m_4$ gives

$$\sum_{i=1}^{4} m_i f_{\mathbf{A}}(\mathrm{e}_i)\eta = \frac{2\kappa\mu}{\kappa+\mu}\cdot$$

$$\begin{bmatrix} 2m_2\eta_{11} + m_3(\eta_{11} + \eta_{22}) & m_3\eta_{12} \\ m_3\eta_{12} & 2m_1\eta_{22} + m_3(\eta_{11} + \eta_{22}) \end{bmatrix}.$$

Inverting the relationship

$$\tau = \left[\sum_{i=1}^{4} m_i f_{\mathbf{A}}(\mathrm{e}_i)\right]\eta,$$

yields

$$\eta = \frac{\kappa+\mu}{4\kappa\mu}\begin{bmatrix} \frac{2m_1\tau_{11} - m_3(\tau_{11}+\tau_{22})}{\Delta} & \frac{\tau_{12}}{m_3} \\ \frac{\tau_{12}}{m_3} & \frac{2m_2\tau_{22} - m_3(\tau_{11}+\tau_{22})}{\Delta} \end{bmatrix}$$

where we have set

$$\Delta = 2m_1m_2 + m_3(m_1 + m_2).$$

This gives explicitly the value of $\mathbf{A}_L^{*-1}\tau$ for any symmetric matrix τ

$$\mathbf{A}_L^{*-1}\tau = \mathbf{A}^{-1}\tau + \frac{1-\theta}{\theta}\eta.$$

We now apply this formula to σ, which is diagonal in the $(\mathbf{n}_1, \mathbf{n}_2)$ basis, and restrict ourselves to the case where $\det(\sigma) > 0$. Equalizing both sides of (22) leads to

$$m_1 = m_2 + \frac{|\sigma_2| - |\sigma_1|}{|\sigma_2| + |\sigma_1|}.$$

Together with the constraints $m_4 = m_3$ and $2m_3 = 1 - (m_1 + m_2)$, it characterizes many possible choices of optimal rank-4 sequential laminate. For simplicity we choose

$$m_1 = \frac{|\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_2 = \frac{2|\sigma_1| - |\sigma_2|}{2(|\sigma_1| + |\sigma_2|)},$$

$$m_3 = m_4 = m_1.$$

For this special choice, we check that $\mathbf{A}_L^*$ is a nondegenerate rank-4 laminate. Introducing $m_0 = m_1(m_1 + 3m_2)$, it is given by

$$\left(\mathbf{A}_L^{*-1}\right)_{1111} = \frac{(\kappa+\mu)(m_0\theta + 3m_1(1-\theta))}{4\kappa\mu\theta m_0},$$

$$\left(\mathbf{A}_L^{*-1}\right)_{1122} = \left(\mathbf{A}_L^{*-1}\right)_{2211} =$$

$$\frac{(\mu-\kappa)m_0\theta + (\kappa+\mu)m_1(\theta-1)}{4\kappa\mu m_0},$$

$$\left(\mathbf{A}_L^{*-1}\right)_{2222} =$$

$$\frac{(\kappa+\mu)(m_0\theta + (2m_2+m_1)(1-\theta))}{4\kappa\mu\theta m_0},$$

$$\left(\mathbf{A}_L^{*-1}\right)_{1212} = \frac{2\kappa\theta m_1 + (1-\theta)(\kappa+\mu)}{4\kappa\mu m_1}.$$

As $m_0, m_1 \neq 0$, all entries of $\mathbf{A}_L^{*-1}$ [and in particular $\left(\mathbf{A}_L^{*-1}\right)_{1212}$] are finite, i.e. the Hooke's law is nondegenerate when $\det \boldsymbol{\sigma} > 0$.

4 Conclusions

In this paper we investigated the optimal microstructures for the Hashin-Shtrikman lower bound on complementary energy which plays a crucial role in the homogenization method for shape and topology optimization. In two space dimension, the well-known optimal rank-2 sequential laminate has the drawback to have a degenerate Hooke's law since it cannot sustain a nonaligned shear stress, and it yields some difficulties in the numerical algorithms deduced from this approach. Here we proved that, for a stress $\boldsymbol{\sigma}$ such that $\det \boldsymbol{\sigma} \leq 0$, any other optimal microstructure is also degenerate. On the contrary, if $\det \boldsymbol{\sigma} > 0$, one can replace the rank-2 sequential laminate by an optimal rank-4 sequential laminate which is not degenerate. Therefore, the aforementioned numerical problems can only be partly alleviated. We do not study the 3-D case, since the optimal rank-3 sequential laminates in 3-D do not suffer from this degeneracy property.

References

Allaire, G.; Kohn, R. 1993a: Optimal bounds on the effective behavior of a mixture of two well-ordered elastic materials. *Quart. Appl. Math.* **51**, 643–674

Allaire, G.; Kohn, R. 1993b: Optimal design for minimum weight and compliance in plane stress using extremal microstructures. *Europ. J. Mech. A/Solids* **12**, 839–878

Allaire, G.; Lods, V. 1997: Minimizers for a double-well problem with affine boundary conditions. *Proc. Roy. Soc. Edin.* (to appear)

Allaire, G.; Bonnetier, E.; Francfort, G.; Jouve, F. 1997: Shape optimization by the homogenization method. *Numerische Mathematik* **76**, 27–68

Ball, J.; James, R. 1987: Fine phase mixtures as minimizers of energy *Arch. Rat. Mech. Anal.* **100**, 13–52

Bendsøe, M.P. *Methods for optimization of structural topology, shape and material.* Berlin, Heidelberg, New York: Springer

Bendsøe, M.P.; Kikuchi, N. 1988: Generating optimal topologies in structural design using a homogenization method. *Comp. Meth. Appl. Mech. Engrg.* **71**, 197–224

Cheng, G.; Olhoff, N. 1981: An investigation concerning optimal design of solid elastic plates. *Int. J. Solids Struct.* **16**, 305–323

Cherkaev, A.V.; Grabovsky, Y.; Movchan, A.B.; Serkov, S.K. 1998: The cavity of the optimal shape under the shear stresses. *Int. J. Solids & Struct.* **35**, 4391–4410

Gibianski, L.; Cherkaev, A. 1997: Design of composite plates of extremal rigidity. In: Cherkaev, A.; Kohn, R. (eds.) *Topics in the mathematical modelling of composite materials. Progress in Nonlinear Differential Equations and their Applications* **31**. Boston: Birkhäuser

Grabovsky, Y.; Kohn, R. 1995a: Microstructures minimizing the energy of a two-phase elastic composite in two space dimensions I: the confocal ellipse construction. *J. Mech. Phys. Solids* **43**, 933–947

Grabovsky, Y.; Kohn, R. 1995b: Microstructures minimizing the energy of a two-phase elastic composite in two space dimensions II: the Vigdergauz microstructure. *J. Mech. Phys. Solids* **43**, 949–972

Francfort, G.; Murat, F. 1996: Homogenization and optimal bounds in linear elasticity. *Arch. Rat. Mech. Anal.* **94**, 307–334

Hashin, Z. 1963: The elastic moduli of heterogeneous materials. *J. Appl. Mech.* **29**, 143–150

Jog, C.; Haber, R.; Bendsøe, M.P. 1994: Topology design with optimized, self-adaptative materials. *Int. J. Numer. Meth. Engrg.* **37**, 1323–1350

Kohn, R. 1991: Relaxation of a double-well energy. *Cont. Mech & Thermodyn.* **3**, 193–236

Kohn, R.; Strang, G. 1986: Optimal Design and Relaxation of Variational Problems I-II-III. *Comm. Pure Appl. Math.* **39**, 113–137, 139–182, 353–377

Murat, F. 1977: Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients. *Ann. Mat. Pura Appl.* **112**, 49–68

Murat, F.; Tartar, L. 1997: Calcul des Variations et Homogénéisation. Les Méthodes de l'Homogénéisation Théorie et Applications en Physique. *Coll. Dir. Etudes et Recherches EDF, Eyrolles*, 319–369. English translation in: Cherkaev, A.; Kohn, R. (eds.) *Topics in the mathematical modelling of composite materials. Progress in Nonlinear Differential Equations and their Applications* **31**. Boston: Birkhäuser

Rozvany, G.; Bendsøe, M.P.; Kirsch, U. 1995: Layout optimization of structures. *Appl. Mech. Rev.* **48**, 41–118

Tartar, L. 1985: Estimation fines de coefficients homogénéisés. In: Krée, P. (ed.) *Ennio Di Giorgi Colloquium. Pitman Research Notes in Math.* **125**, 168–187

Tartar, L. 1990: H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. *Proc. Roy. Soc. Edin.* **115A**, 193–230

Vigdergauz, S. 1994: Two-dimensional grained composites of extreme rigidity. *ASME J. Appl. Mech.* **61**, 390–394

Received May 12, 1998
Revised manuscript received October 5, 1998