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Multiscale convergence and reiterated homogenisation

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This paper generalises the notion of two-scale convergence to the case of multiple separated scales of periodic oscillations. It allows us to introduce a multi-scale convergence method for the reiterated homogenisation of partial differential equations with oscillating coefficients. This new method is applied to a model problem with a finite or infinite number of microscopic scales, namely the homogenisation of the heat equation in a composite material. Finally, it is generalised to handle the homogenisation of the Neumann problem in a perforated domain.

1. Introduction

This paper is a contribution to the theory of homogenisation for partial differential equations (p.d.e.). In many fields of physics, mechanics, or engineering sciences, physical phenomena occur in highly heterogeneous media, the properties of which vary on many different length scales. Quite often these phenomena are correctly modelled by a set of p.d.e. at some microscopic level (where there is no heterogeneity), while the relevant quantities or behaviours which the physicist or engineer wants to measure or evaluate are intrinsically macroscopic. Between these two levels of description, there may be several orders of magnitude and very complicated patterns or hierarchies of heterogeneities. Therefore, the direct numerical computation and prediction of effective macroscopic quantities can be very costly or even out of reach. In this case, it is preferable to analyse further the available microscopic models and deduce, by some averaging or asymptotic process, suitable 'homogenised' macroscopic laws.

This is precisely the purpose of the mathematical theory of homogenisation. Three branches of this theory can be distinguished: the most general is that of G or H-convergence, which places no restriction on the size or arrangement of the heterogeneities (see e.g. [**22, 26, 27, 29**]); the second one deals with probabilistic and stochastic descriptions of heterogeneous media (see e.g. [**8, 20, 24**]); the third one is devoted to periodic structures (see e.g. [**5, 7, 25**]). Although the latter approach is certainly the less general one, since periodicity is a very strong assumption not always encountered in real media, it has acquired a paramount importance for at least three reasons. First of all, it is the easiest to work with, thanks to the celebrated two-scale asymptotic expansions method (consequently, it is the most well-known among non-mathematicians). Secondly, it can handle very complicated models which are not amenable to the other methods. Last, but not least, its importance goes far beyond periodic materials since, in some cases, it has been proved that there is no loss of generality in considering only periodic media (for example, in the study of effective properties of composite materials, see the theorems on 'the local character of G-closure' in [**14**, **17**]). There is therefore a considerable amount of literature concerning homogenisation in periodic structures (for example, see the formidable bibliography in [**16**]). The present paper pertains to that 'periodic' approach of homogenisation.

Let us describe more precisely a model problem in this framework. For example, we consider a conduction problem in composite material which, of course, is assumed to have a periodic structure. Denoting by ε the microstructure lengthscale, and by $Y=[0,1]^N$ the reference unit cell, our composite material occupies a bounded domain Ω in $\mathbb{R}^N$ and has a periodicity of εY. In other words, its conductivity tensor $A_\varepsilon(x)$, describing its pointwise structure, is given by

$$A\left(x, \frac{x}{\varepsilon}\right) \quad \text{for any } x \in \Omega, \tag{1.1}$$

where $A(x, y)$ is the Y-periodic, with respect to the variable y, conductivity tensor in the unit cell Y. The conduction problem is then to find the potential u_ε solution of

$$\begin{cases} -\operatorname{div} A_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f \in L^2(\Omega)$ is a given source term. Under a standard hypothesis on the tensor A, this problem is known to have a unique solution in $H_0^1(\Omega)$. Its homogenisation consists in an asymptotic analysis of (1.2) as the parameter ε goes to zero. The sequence of solutions u_ε is easily seen to be bounded in $H_0^1(\Omega)$. Thus, up to a subsequence, it converges weakly to some limit u. The question is to find which homogenised equation is satisfied by u.

The answer can be obtained by means of formal asymptotic expansions, but recently a new method, called 'two-scale convergence' has been introduced by Allaire [**2**] and Nguetseng [**23**] which, in some sense, is the mathematically rigorous version of this 'ansatz' method. The name of this method makes reference to the two natural lengthscales in the model problem: the macroscopic one, corresponding to the x variable in the domain Ω, and the microscopic one associated to the y variable in the reference period Y. The very fact that there are only two lengthscales in this problem has been enforced by the modelling, somehow arbitrarily. Indeed, there may well be several microscopic scales instead of just one. For example, just think of a composite medium made of different types of inclusions in a matrix material, having different sizes and different periodicities.

Thus, it is very natural to extend the above model to the more realistic situations where several microscopic scales have to be taken into account (the intermediate scales are sometimes called mesoscales). More precisely, denoting by $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n$ a set of n ordered lengthscales, which all depend on a single parameter ε, we consider the same conduction problem (1.2) where the conductivity tensor is now defined by

$$A_\varepsilon(x) = A\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right), \tag{1.3}$$

where $A(x, y_1, \ldots, y_n)$ is Y-periodic with respect to each variable y_k. Of course, each of these scales is microscopic in the sense that we have

$$\lim_{\varepsilon \to \infty} \varepsilon_k = 0 \quad \text{for } 1 \leqq k \leqq n, \tag{1.4}$$

but we also make a fundamental hypothesis on the separation of scales, namely

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \text{for } 1 \leqq k \leqq n-1. \tag{1.5}$$

This means that each scale can be distinguished from the others, i.e. they are not of the same order of magnitude. The purpose of this paper is to generalise the two-scale convergence method, developed in [**2**] and [**23**], to the case of reiterated homogenisation problems (in the terminology of [**7**]), for the homogenisation of p.d.e. whose coefficients oscillate periodically on *several* scales (a model problem is precisely equation (1.2) with tensor (1.3)). The main goal of this paper is to provide the main tools of our new multiscale convergence method which is applied to the above model problem and to a similar problem stated in a periodically perforated porous medium. Of course, our method could be equally applied to other problems of greater interest from a physical point of view! We leave this task to some future work, and we focus here on the mathematical foundations of the method rather than its applications. Before introducing our main results, we briefly review the state of the art on this problem. In the case where the scales ε_k, $1 \leqq k \leqq n$, are successive powers of the parameter ε, i.e.

$$\varepsilon_k = \varepsilon^k \quad \text{for } 1 \leqq k \leqq n, \tag{1.6}$$

it is not difficult, at least formally, to homogenise problem (1.2)–(1.3) by means of multiple-scale asymptotic expansions. The rigorous justification of this process is somehow more delicate. It was done by Bensoussan, Lions and Papanicolaou [**7**], in the simpler case of two microscopic scales, and it is quite technical since it involves Meyers' Theorem (see the second subsection of Section 2 below for a more complete discussion). When the scales do not satisfy condition (1.6), the generalisation of both the asymptotic expansions and the convergence proof is not obvious at all. Some partial results have been obtained by Murat (unpublished), but the general case was still an open problem. The multiscale convergence method allows us definitely to solve this question, while keeping the technicalities to a minimum.

Let us now introduce the notion of multiscale convergence and the basic compactness result:

THEOREM 1.1. *Let u_ε be a bounded sequence in $L^2(\Omega)$. There exist a subsequence (still denoted by u_ε) and a function $u_0(x, y_1, \ldots, y_n)$ in $L^2(\Omega \times Y_1 \times \ldots \times Y_n)$ such that*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right) dx$$

$$= \int_\Omega \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \ldots, y_n)\varphi(x, y_1, \ldots, y_n)\, dx\, dy_1 \ldots dy_n \tag{1.7}$$

for any smooth function $\varphi(x, y_1, \ldots, y_n)$ which is Y-periodic for all variables y_k. Such a sequence is said to multiscale converge to u_0.

Theorem 1.1 is a straightforward generalisation of the corresponding result for the two-scale convergence. However, the following theorem is not so obvious, and its proof is the most difficult (if not important) result of this paper.

THEOREM 1.2. *Let u_ε be a bounded sequence in $H^1(\Omega)$. Up to a subsequence, there exists a function $u(x)$ in $H^1(\Omega)$ and n functions $u_k(x, y_1, \ldots, y_k)$ in $L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y_k)]$ such that u_ε multiscale converges to u and ∇u_ε to*

$$\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k).$$

Of course, $u(x)$ is also the usual limit in $H^1(\Omega)$ of the sequence u_ε. The other terms in the multiscale limit of ∇u_ε can be interpreted as being the gradient limits at each scale.

With these results, the homogenisation of problem (1.2)–(1.3) becomes an easy task, and we shall prove:

THEOREM 1.3. *The sequence u_ε of solutions of* (1.2)–(1.3) *weakly converges in $H^1_0(\Omega)$ to the unique solution u of the homogenised problem*

$$\begin{cases} -\operatorname{div} A^* \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where the homogenised matrix A^* is computed by homogenising separately and successively the different scales, starting from the smallest one ε_n up to the largest one ε_1 (for details, see Corollary 2.12).

Furthermore, under an additional assumption on the smoothness of the matrix A, we have

$$\left[u_\varepsilon(x) - u(x) - \sum_{k=1}^{n} \varepsilon_k u_k\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_k}\right)\right] \to 0 \quad \text{in } H^1(\Omega) \text{ strongly}, \tag{1.9}$$

where the functions u_k, $1 \leq k \leq n$, are the components of the multiscale limit of ∇u_ε.

In some sense formula (1.9) gives the leading term of the multiscale asymptotic expansion of the solution u_ε. Note that the usual rule of such ansatz suggests that we should include other terms of the type ε_k^p which are smaller than ε_n, but this is not actually necessary.

Furthermore, following an idea of Bensoussan and Lions [**6**], we can generalise Theorem 1.3 to the case where the matrix $A_\varepsilon(x)$ has an infinite number of oscillating arguments (see the third subsection of Section 2, below).

To complement our study of the model problem (1.2)–(1.3), we consider a similar problem in a porous medium, i.e. we replace the domain Ω by a multiscale periodically perforated domain Ω_ε. Actually, homogenisation in porous media (modelled by a perforated domain) is a problem of paramount importance with many applications in geophysics or petroleum engineering, and it is another motivation of the present work. Of course, periodicity is a very crude assumption for modelling porous media, but our approach has, at least, the advantage of taking in account several lengthscales of pores (or fractures) as is frequently the case for real porous media. More precisely, we study the following Neumann problem:

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.10}$$

where the perforated domain Ω_ε is defined by its characteristic function $\chi_\varepsilon(x)$ given by

$$\chi_\varepsilon(x) = \chi(x)\chi_1\left(\frac{x}{\varepsilon_1}\right)\dots\chi_n\left(\frac{x}{\varepsilon_n}\right),$$

where $\chi(x)$ is the characteristic function of a bounded domain Ω, $(\chi_k(y_k))_{1\leqq k\leqq n}$ is a family of Y-periodic characteristic functions which corresponds to a family of 'patterns' $(Y_k^*)_{1\leqq k\leqq n}$ in the unit cell Y. We assume that for any scale $1\leqq k\leqq n$, the pattern Y_k^*, extended by Y-periodicity in $\mathbf{R}^N$, yields a connected 'material' domain. However, the 'holes' (i.e. $Y_k\backslash Y_k^*$) can be either connected or not. We also assume that the scales $(\varepsilon_k)_{1\leqq k\leqq n}$ satisfy assumptions (1.4)–(1.5), and that they are 'well-separated', namely there exists a positive integer m such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^m = 0 \quad \text{for } 1\leqq k\leqq n-1. \tag{1.11}$$

(Assumption (1.11) is stronger than (1.5): see Definition 3.1 for additional comments.)

A well-known difficulty for the homgenisation of equation (1.10) is that the only available *a priori* estimate is

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqq c, \tag{1.12}$$

where c is a constant independent of ε. The point is that (1.12) does not imply that the sequence u_ε is bounded in a fixed Sobolev space, and thus we cannot extract a weakly converging subsequence from it. To bypass this difficulty, a possible trick, due to Tartar (see [**27**]), is to build an extension operator from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$ such that the extended sequence u_ε would be bounded in $H^1(\Omega)$. This is possible in the case of a single microscopic scale (see [**1, 15**]) but it seems to be very delicate to extend this result to the present situation where there are several microscopic scales. However, as was recognised in [**2**], the two-scale convergence method does not require such extension techniques for the homogenisation of (1.10). Therefore, without using any extension operator (apart from the trivial extension by 0 in the holes $\Omega\backslash\Omega_\varepsilon$), we shall prove in Section 4 the following theorem:

THEOREM 1.4. *Denoting by* $\tilde{\ }$ *the extension by zero in the holes* $\Omega \backslash \Omega_\varepsilon$, *and by* θ *the material volume fraction, i.e.*

$$\theta = \prod_{k=1}^{n} \fint_Y \chi_k(y_k)\, dy_k,$$

the sequence $\tilde{u}_\varepsilon$ *converges weakly in* $L^2(\Omega)$ *to* θu, *where* $u(x)$ *is the unique solution in* $H^1(\Omega)$ *of the homogenised problem*

$$\begin{cases} -\operatorname{div} A^* \nabla u + \theta u = \theta f & \text{in } \Omega, \\ (A^* \nabla u) \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where the homogenised matrix A^* *is computed by homogenising separately and successively the different patterns* Y_k^* *from the smallest one* $k = n$ *to the largest one* $k = 1$.

Let us now proceed to a brief description of the contents of this paper. The main results of multiscale convergence are stated in the first subsection of Section 2, while the second subsection is devoted to the application of these results to the homogenisation of the model problem (1.2)–(1.3), and the third subsection investigates the case of an infinite number of scales. Section 3 focuses on the proof of Theorem 1.2. In the first subsection of Section 3, a first simplified proof is given in the case of well-separated scales (see (1.11)). The general case is treated in detail in the second subsection. Finally, Section 4 generalises the previous results to the case of periodically perforated domains and furnishes a proof of Theorem 1.4 above.

2. Multiscale convergence

Main results

This subsection contains the definition and the properties of the multiple-scale convergence which generalises the previous notion of two-scale convergence, introduced by Allaire [2] and Nguetseng [23]. In the following, we denote by:

Ω a bounded open set of $\mathbf{R}^N$ ($N \geq 1$);

n the number of scales, a positive integer;

$\varepsilon_1, \ldots, \varepsilon_n$ n positive functions of $\varepsilon > 0$ which converge to 0 as ε does;

$Y_1, \ldots, Y_n$ n copies of the unit cube $[0, 1]^N$;

$C_\#(Y_1 \times \ldots \times Y_n)$ the space of continuous functions $\varphi(y_1, \ldots, y_n)$ which are Y_k-periodic with respect to all its variables y_k, for $k \in \{1, \ldots, , n\}$;

$H^1_\#(Y_k)$ the space of functions $\varphi(y_k)$ in $H^1_{\mathrm{loc}}(\mathbf{R}^N)$ which are Y_k-periodic.

Throughout this paper, we assume that the scales are ordered in such a way that ε_n is the smallest and ε_1 is the largest one:

ASSUMPTION 3.1. The functions $\varepsilon_1, \ldots, \varepsilon_n$ are assumed to be separated, i.e. they satisfy

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \forall k \in \{1, \ldots, n-1\}.$$

DEFINITION 2.2. For any Y_k-periodic function (for all $k \in \{1, \ldots, n\}$) $\varphi(x, y_1, \ldots, y_n)$, the oscillating function $[\varphi]_\varepsilon$ is defined by:

$$[\varphi]_\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right).$$

Thanks to Assumption 2.1 on the separation of scales, it is easily seen that, at least four smooth functions φ, the function $[\varphi]_\varepsilon(x)$ will converge to its average

$$\int_{Y_1} \cdots \int_{Y_n} \varphi(x, y_1, \ldots, y_n)\, dy_1 \ldots dy_n,$$

in the sense of distributions in Ω. This basic property of the oscillating functions defined above allows us to introduce the next definition.

DEFINITION 2.3. A sequence u_ε of $L^2(\Omega)$ is said to $(n+1)$-scale converge to $u_0(x, y_1, \ldots, y_n) \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$ if and only if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)[\varphi]_\varepsilon(x)\, dx$$

$$= \int_\Omega \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \ldots, y_n)\varphi(x, y_1, \ldots, y_n)\, dx\, dy_1 \ldots dy_n,$$

for any function $\varphi \in L^2[\Omega; C_\#(Y_1 \times \ldots \times Y_n)]$. We denote this convergence by

$$u_\varepsilon \xrightarrow{(n+1)\text{-scale}} u_0(x, y_1, \ldots, y_n).$$

As for the two-scale convergence, this definition makes sense because of the following compactness theorem:

THEOREM 2.4. *Under Assumption 2.1 of separation of scales, from each bounded sequence in $L^2(\Omega)$ one can extract a subsequence which $(n+1)$-scale converges to a limit $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$.*

We also have a corrector result which involves a strong convergence in $L^2(\Omega)$.

THEOREM 2.5. *Let u_ε be a sequence of functions in $L^2(\Omega)$ with $(n+1)$-scale converges to $u_0(x, y_1, \ldots, y_n)$ and which satisfies*

$$\lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega \times Y_1 \times \ldots \times Y_n)}.$$

Then, for any sequence v_ε which $(n+1)$-scale converges to $v_0(x, y_1, \ldots, y_n)$, one has

$$u_\varepsilon(x)v_\varepsilon(x) \rightharpoonup \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \ldots, y_n)v_0(x, y_1, \ldots, y_n)\, dy_1 \ldots dy_n \quad \textit{weakly in } L^1(\Omega).$$

Furthermore, if $u_0(x, y_1, \ldots, y_n)$ belongs to $L^2[\Omega; C_\#(Y_1 \times \ldots \times Y_n)]$, one has

$$\lim_{\varepsilon \to 0} \| u_\varepsilon - [u_0]_\varepsilon \|_{L^2(\Omega)} = 0.$$

The proofs of Theorems 2.4 and 2.5 are simple adaptations of [**2**, Theorem 1.2 and Theorem 1.8] and thus do not deserve to be repeated here. For the sake of the nonexpert reader, we content ourselves with indicating the key ideas of the proof of

Theorem 2.4. The first point is to recognise that the integral $\int_\Omega u_\varepsilon[\varphi]_\varepsilon\,dx$ is a linear form on the space of test functions $L^2[\Omega; C_\#(Y_1 \times \ldots \times Y_n)]$. Thus it can be identified to a duality product $\langle \mu_\varepsilon, \varphi\rangle$ where μ_ε is a bounded sequence of measures in the dual space of $L^2[\Omega; C_\#(Y_1 \times \ldots \times Y_n)]$. Since this space is separable, one can extract a weakly convergent sequence to a limit measure μ_0. The second key point is to check that $\langle \mu_0, \varphi\rangle$ is a linear form on $L^2(\Omega \times Y_1 \times \ldots \times Y_n)$, yielding that μ_0 is indeed a function $u_0(x, y_1, \ldots, y_n)$ in $L^2(\Omega \times Y_1 \times \ldots \times Y_n)$. This is due to the following well-known convergence of oscillating functions (see e.g. [18]):

$$\int_\Omega ([\varphi]_\varepsilon(x))^2\,dx \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} (\varphi(x, y_1, \ldots, y_n))^2\,dx\,dy_1 \ldots dy_n$$

for any $\varphi \in L^2[\Omega; C_\#(Y_1 \times \ldots \times Y_n)]$.

The next theorem, which investigates the case of bounded sequences in $H^1(\Omega)$, is of paramount importance for applications to homogenisation problems (see the next subsection).

THEOREM 2.6. *For any bounded sequence u_ε in $H^1(\Omega)$, there exists a function $u(x) \in H^1(\Omega)$ and n functions $u_k(x, y_1, \ldots, y_k) \in L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y_k)]$ such that, up to a subsequence,*

$$u_\varepsilon \xrightarrow{(n+1)\text{-scale}} u(x),$$

$$\nabla u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k).$$

Furthermore, any such $(n+1)$-scale limit $(u, u_1, \ldots, u_n)$ is attained by a bounded sequence u_ε in $H^1(\Omega)$.

THEOREM 2.7. *Let ξ_ε be a bounded sequence in $L^2(\Omega)^N$ which $(n+1)$-scale converges to a limit $\xi_0(x, y_1, \ldots, y_n)$ in $L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)^N]$. Assume that ξ_ε is divergence-free, i.e.*

$$\operatorname{div} \xi_\varepsilon = 0 \quad \text{in } \Omega.$$

Then, the limit ξ_0 satisfies the 'generalised' divergence-free condition

$$\begin{cases} \operatorname{div}_{y_n} \xi_0 = 0, \\ \displaystyle\int_{Y_{k+1}} \cdots \int_{Y_n} \operatorname{div}_{y_k} \xi_0\,dy_{k+1} \ldots dy_n = 0 \qquad 1 \leqq k \leqq n-1 \\ \displaystyle\int_{Y_1} \cdots \int_{Y_n} \operatorname{div}_x \xi_0\,dy_1 \ldots dy_n = 0. \end{cases}$$

Furthermore, any function $\xi_0(x, y_1, \ldots, y_n)$ in $L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)^N]$ which satisfies the 'generalised' divergence-free condition is attained as the $(n+1)$-scale limit of a divergence-free bounded sequence ξ_ε in $L^2(\Omega)^N$.

REMARK 2.8. There is a subtle point in Theorem 2.7 concerning the periodicity of a function ξ_0 satisfying the 'generalised' divergence-free condition. This is reflected in our meaningless notation $\xi_0(x, y_1, \ldots, y_n) \in L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)^N]$. Indeed, as is well known, L^2 functions do not have trace properties, and the space $L^2_\#(Y_k)$ coincides with the usual space $L^2(Y_k)$ in which functions are extended by periodicity. However, a divergence-free function $\Psi \in L^2(Y_k)^N$ has a normal trace $\Psi \cdot \nu$ in $H^{-1/2}(\partial Y_k)$. Thus, the normal components of the vector field ξ_0 must satisfy some type of periodicity conditions. These conditions are incluced in the 'generalised' divergence-free condition above if equalities are taken in the sense of distributions in $\mathbf{R}^N$, or equivalently

$$\int_{Y_k} \dots \int_{Y_n} \xi_0 \cdot \nabla_{y_k} \varphi \, dy_k \dots dy_n = 0 \quad \text{for any } \varphi \in H^1_\#(Y_k) \text{ and } 1 \leqq k \leqq n.$$

More precisely, they are

$$\begin{cases} \xi_0 \cdot \nu & \text{takes opposite values on opposite faces of } Y_n, \\ \displaystyle\int_{Y_{k+1}} \dots \int_{Y_n} \xi_0 \cdot \nu & \text{takes opposite values on opposite faces of } Y_k,\ 1 \leqq k \leqq n-1. \end{cases}$$

REMARK 2.9. Contrary to Theorems 2.4 and 2.5, Theorem 2.6 is not a simple generalisation of the equivalent results available in the two-scale case (see [**2**, Proposition 1.14]). The proof of Theorem 2.6 is very delicate due to the possible interactions between the different oscillating scales. This proof is the focus of Section 3. In the first subsection, an elementary proof of Theorem 2.6 is given under an additional assumption on the scales (see Definition 3.1 on *well-separated* scales). In the second subsection, taking advantage of the insight into the problem provided by the first subsection, we complete the proof of Theorem 2.6 for any type of simply separated scales.

REMARK 2.10. Theorem 2.7 is a key ingredient for the homogenisation of Stokes equations describing the flow of an incompressible fluid. However, this problem is not addressed in the present paper. Thus, our main motivation for stating this result is the complementarity of Theorems 2.6 and 2.7. As is well known, 'gradients are orthogonal to divergence-free fields', and in Lemma 3.7 below, we shall prove a similar result for $(n+1)$-scale limits. In other words, the 'generalised' gradients of Theorem 2.6 are orthogonal in $L^2[\Omega; L^2_\#(Y_1 \times \dots \times Y_n)^N]$ to the 'generalised' divergence-free fields of Theorem 2.7. This describes completely the set of possible $(n+1)$-scale limits.

Application to the multiscale homogenisation

We immediately apply the results of the previous subsection to a model problem in homogenisation theory. We consider a conduction problem in a multiply-periodic domain.

The conductivity tensor of this domain is a matrix-function $[A]_\varepsilon(x)$, where $A(x, y_1, \dots, y_n)$ is Y_k-periodic for all $k \in \{1, \dots, n\}$, not necessarily symmetric, and satisfies for some $\beta \geqq \alpha > 0$

$$\alpha|\xi|^2 \leqq A(x, y_1, \dots, y_n)\xi \cdot \xi \leqq \beta|\xi|^2 \quad \text{a.e. in } \Omega \times Y_1 \times \dots \times Y_n \text{ for any } \xi \in \mathbf{R}^N. \tag{2.1}$$

Since assumption (2.1) does not guarantee that the function $[A]_\varepsilon$ is measurable in Ω, we add the following assumption:

$$[A]_\varepsilon \in L^\infty(\Omega)^{N^2}. \tag{2.2}$$

For a given source term $f \in L^2(\Omega)$, we consider the following linear second-order elliptic equation

$$\begin{cases} -\operatorname{div} [A]_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

which, as is well-known, admits a unique solution $u_\varepsilon \in H_0^1(\Omega)$. When the parameter ε goes to 0 (i.e. the different scales $\varepsilon_1, \ldots, \varepsilon_n$ go to zero), it is easily seen that the sequence of solutions u_ε remains bounded in $H_0^1(\Omega)$. The homogenisation problem for (2.3) is to investigate what is the limit (or homogenised) equation satisfied by the limit of the sequence u_ε (if any).

This problem has already been investigated by Bensoussan, Lions and Papanicolaou in [**7**], where it is called a *reiterated homogenisation problem* (see [**7**, Section 8, Chapter I]). Let us briefly describe their method: as a first step, they carry on the homogenisation of (2.3) for a sequence of smooth coefficients matrices A_δ which approximate the original matrix A as δ goes to zero; in a second step, they show that one can pass to the limit with respect to δ in the homogenisation process.

The main tool of their first step is the so-called energy method of Tartar [**27**], i.e. they consider a family of test functions w_ε such that $[A_\delta]_\varepsilon \nabla w_\varepsilon$ has a compact divergence in $H^{-1}(\Omega)$. They obtain such functions by using an asymptotic development based on scales of the type $\varepsilon_k = \varepsilon^k$ for all $k \in \{1, \ldots, n\}$. Remarking that the homogenisation process with respect to a single scale depends continuously on the other variables (corresponding to the other scales), they obtained the homogenised problem for (2.3) by successively homogenising the different scales, from the smallest to the largest one. The main difficulty in their second step comes from the fact that the original matrix is not continuous in general, but simply bounded. Since continuous functions are not dense in L^∞, one needs to estimate the difference $(A - A_\delta)$ in some L^p-norm for a finite p. Thus, to control the energy integral

$$\int_\Omega [A]_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx,$$

a better estimate than $u_\varepsilon \in H_0^1(\Omega)$ is required. For this reason, a key ingredient in their proof is Meyers' Theorem [**21**], as generalised in [**7**, Section 4, Chapter I].

The method of [**7**] is thus very technical and, furthermore, is restricted to the following situation: the scales are successive powers of ε (i.e. $\varepsilon_k = \varepsilon^k$), and the matrix A is continuous at all scales but one. For two oscillating scales, Murat has extended their method to the case of ε_1 and ε_2 being well-separated (a stronger assumption than Assumption 2.1, see Definition 3.1) by using a clever asymptotic expansion (unpublished). It seems difficult (or, at least, very tedious) to extend this method to more general situations. On the contrary, the multiscale convergence method covers a wide range of situations while keeping the technicalities to a strict minimum. In particular, we will not use Meyers' Theorem in the sequel. We recall that, as far as the scales are separated (see Assumption 2.1), their choice is completely free. Concerning the matrix A, our main results hold in many different situations (see Remark 2.13).

THEOREM 2.11. *Under Assumption 2.1 on the separation of scales, and assuming further that the matrix A satisfies* (2.1), (2.2), *and the following condition:*

$$[A]_\varepsilon \xrightarrow{(n+1)\text{-scale}} A \quad \text{and} \quad \lim_{\varepsilon\to 0} \|([A]_\varepsilon)_{ij}\|_{L^2(\Omega)} = \|A_{ij}\|_{L^2(\Omega\times Y_1\times\ldots Y_n)}, \tag{2.4}$$

the solution u_ε of the conduction problem (2.3) *converges weakly to a function u of $H_0^1(\Omega)$ and its gradient ∇u_ε $(n+1)$-scale converges to a limit*

$$\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k),$$

where $(u, u_1, \ldots, u_n)$ is the unique solution in the space

$$V = H_0^1(\Omega) \times \prod_{k=1}^{n} L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H_\#^1(Y_k)/\mathbf{R}] \tag{2.5}$$

of the so-called $(n+1)$-scale homogenised system (composed of $(n+1)$ p.d.e.)

$$\begin{cases} -\operatorname{div}_{y_n} A\left(\nabla u(x) + \sum_{j=1}^{n} \nabla_{y_j} u_j\right) = 0, \\ -\operatorname{div}_{y_n}\left[\int_{Y_{k+1}} \ldots \int_{Y_n} A\left(\nabla u(x) + \sum_{j=1}^{n} \nabla_{y_j} u_j\right) dy_{k+1} \ldots dy_n\right] = 0 \qquad 1 \leqq k \leqq n-1, \\ -\operatorname{div}_x\left[\int_{Y_1} \ldots \int_{Y_n} A\left(\nabla u(x) + \sum_{j=1}^{n} \nabla_{y_j} u_j\right) dy_1 \ldots dy_n\right] = f. \end{cases} \tag{2.6}$$

COROLLARY 2.12. *The limit u of the sequence u_ε is also the unique solution of the 'usual' limit equation in $H_0^1(\Omega)$*

$$\begin{cases} -\operatorname{div} A^* \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{2.7}$$

where the matrix $A^* = A_0^*$ is defined by the inductive homogenisation formulae

$$\begin{cases} A_n^* = A(x, y_1, \ldots, y_n), \\ A_k^* = A_k^*(x, y_1, \ldots, y_k) \\ \qquad \text{obtained by periodic homogenisation of } A_{k+1}^*\left(x, y_1, \ldots, y_k, \dfrac{z}{\varepsilon}\right), \\ A_0^* = A_0^*(x) \text{ obtained by periodic homogenisation of } A_1^*\left(x, \dfrac{z}{\varepsilon}\right); \end{cases} \tag{2.8}$$

in other words:

$$A_k^* \xi = \int_{Y_{k+1}} A_{k+1}^*(\xi + \nabla_{y_{k+1}} w_{k+1}^\xi)\, dy_{k+1}$$

for all vector ξ, with $w_{k+1}^\xi \in L^2[\Omega \times Y_1 \times \ldots \times Y_k; H_\#^1(Y_{k+1})/\mathbf{R}]$ solution of

$$\begin{cases} \operatorname{div}_{y_{k+1}}[A_{k+1}^*(\xi + \nabla_{y_{k+1}} w_{k+1}^\xi)] = 0 & \text{in } Y_{k+1}, \\ w_{k+1}^\xi \quad Y_{k+1}\text{-periodic}. \end{cases}$$

REMARK 2.13. (1) We do not know of any explicit characterisation of the matrices A which satisfy assumption (2.4) in Theorem 2.11. However, apart from A being a smoth matrix-function, we know several sufficient conditions for (2.4) to hold:

(i) $A \in L^\infty[\Omega; C_\#(Y_1 \times Y_2 \times \ldots \times Y_n)]^{N^2}$;

(ii) $A \in L^\infty[Y_k; C_\#(\Omega \times Y_1 \times \ldots \times Y_{k-1} \times Y_{k+1} \times \ldots \times Y_n)]^{N^2}$ for $1 \leqq k \leqq n$;

(iii) $\begin{cases} \text{the entries of } A \text{ are finite sums of products of the type} \\ \varphi_0(x) \prod_{k=1}^{n} \varphi_k(y_k) \quad \text{with} \quad \varphi_0 \in L^\infty(\Omega) \quad \text{and} \quad \varphi_k \in L^\infty_\#(Y_k). \end{cases}$

(2) Let us emphasise that Corollary 2.12 shows that the homogenised matrix-function A^* is obtained by reiteration of n periodic homogenisation problems, successively from the smallest to the largest scale, as already proved in [7].

The following result is a consequence of Theorem 2.5 on corrector results for $(n+1)$-scale convergence.

THEOREM 2.14. *Assume that the solution $(u, u_1, \ldots, u_n)$ of the $(n+1)$-scale homogenised problem* (2.6) *is smooth, say $u_k \in L^2[\Omega; C^1_\#(Y_1 \times \ldots \times Y_k)]$ for all $k \in \{1, \ldots, n\}$. Then, one has the following corrector result*:

$$\left[u_\varepsilon(x) - u(x) - \sum_{k=1}^{n} \varepsilon_k u_k\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_k}\right)\right] \to 0 \quad \textit{strongly in } H^1(\Omega).$$

Proof of Theorem 2.11. From Theorem 2.6, the following convergences hold true, up to a subsequence,

$$u_\varepsilon \xrightarrow{(n+1)\text{-scale}} u(x),$$

$$\nabla u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k),$$

where $(u, u_1, \ldots, u_n) \in V$ defined by (2.5).

Let $\varphi \in \mathscr{D}(\Omega)$ and $\varphi_k \in \mathscr{D}[\Omega: C^\infty_\#(Y_1 \times \ldots \times Y_k)]$ for $1 \leqq k \leqq n$. Plugging the test function

$$\varphi + \sum_{k=1}^{n} \varepsilon_k [\varphi_k]_\varepsilon$$

into the variational formulation of problem (2.3) gives (with the notation $\varepsilon_0 = 1$, $y_0 = x$)

$$\int_\Omega [A]_\varepsilon \nabla u_\varepsilon \cdot \nabla\left(\varphi + \sum_{k=1}^{n} \varepsilon_k [\varphi_k]_\varepsilon\right) dx = \int_\Omega [A]_\varepsilon \nabla u_\varepsilon \cdot \left(\nabla_x \varphi + \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{\varepsilon_k}{\varepsilon_j} [\nabla_{y_j} \varphi_k]_\varepsilon\right) dx$$
$$= \int_\Omega f\left(\varphi + \sum_{k=1}^{n} \varepsilon_k [\varphi_k]_\varepsilon\right) dx.$$

Owing to the condition (2.4) satisfied by $[A]_\varepsilon$ and Theorem 2.5, the $(n+1)$-scale convergence applied to the previous equality gives

$$\int_\Omega \int_{Y_1} \cdots \int_{Y_n} A\left(\nabla_x u + \sum_{k=1}^{n} \nabla_{y_k} u_k\right) \cdot \left(\nabla_x \varphi + \sum_{k=1}^{n} \nabla_{y_k} \varphi_k\right) dx\, dy_1 \ldots dy_n$$

$$= \int_\Omega \int_{Y_1} \cdots \int_{Y_n} f\varphi \, dx\, dy_1 \ldots dy_n = \int_\Omega f\varphi \, dx. \tag{2.9}$$

By density, the equality (2.9) holds for any $(\varphi, \varphi_1, \ldots, \varphi_n) \in V$. Since A is coercive by (2.1), the bilinear form defined by the left-hand side of (2.9) is coercive on V endowed with the norm

$$\| \nabla \varphi \|_{L^2(\Omega)} + \sum_{k=1}^{n} \| \nabla_{y_k} \varphi_k \|_{L^2(\Omega \times Y_1 \times \ldots \times Y_k)}.$$

By the Lax–Milgram Theorem, there exists a unique solution $(u, u_1, \ldots, u_n)$ of (2.9) in V. Thus the entire sequences u_ε and ∇u_ε converge to their limits. Finally, an easy calculation shows that the variational formulation (2.9) is equivalent to the system of equations (2.6), which hence concludes the proof. □

Proof of Corollary 2.12. Let $(e_1, e_2, \ldots, e_N)$ be a basis of $\mathbb{R}^N$. From the $(n+1)$-scale homogenised system (2.6), one can isolate an equation in Y_n for the unknown function u_n:

$$\begin{cases} -\operatorname{div}_{y_n} A\left(\nabla_x u + \sum_{k=1}^{n} \nabla_{y_k} u_k\right) = 0 \quad \text{in } Y_n, \\ u_n \text{ is } Y_n\text{-periodic with respect to } y_n. \end{cases}$$

Thus, u_n can be computed in terms of the other unknowns $(u, u_1, \ldots, u_{n-1})$:

$$u_n(x, y_1, \ldots, y_n) = \sum_{i=1}^{N} w_i(x, y_1, \ldots, y_n) \left(\frac{\partial u}{\partial x_i} + \sum_{j=1}^{n-1} \frac{\partial u_j}{\partial y_{ji}}\right), \tag{2.10}$$

where each function $w_i \in L^\infty[\Omega \times Y_1 \times \ldots \times Y_{n-1}; H^1_\#(Y_n)/\mathbb{R}]$ is the solution of the cell problem in Y_n

$$\begin{cases} -\operatorname{div}_{y_n} A(e_i + \nabla_{y_n} w_i) = 0 \quad \text{in } Y_n \\ y_n \mapsto w_i \quad Y_n\text{-periodic} \end{cases} \qquad 1 \leq i \leq N.$$

Replacing u_n by its expression (2.10) in the homogenised system (2.6) and averaging on Y_n, we obtain a similar system with one scale less:

$$\begin{cases} -\operatorname{div}_{y_{n-1}} A^*_{n-1}\left(\nabla_x u + \sum_{j=1}^{n-1} \nabla_{y_j} u_j\right) = 0, \\ -\operatorname{div}_{y_k}\left[\int_{Y_{k+1}} \cdots \int_{Y_{n-1}} A^*_{n-1}\left(\nabla_x u + \sum_{j=1}^{n-1} \nabla_{y_j} u_j\right) dy_{k+1} \ldots dy_{n-1}\right] = 0, 1 \leq k \leq n-2, \\ -\operatorname{div}_x\left[\int_{Y_1} \cdots \int_{Y_{n-1}} A^*_{n-1}\left(\nabla_x u + \sum_{j=1}^{n-1} \nabla_{y_j} u_j\right) dx\, dy_1 \ldots dy_{n-1}\right] = f, \end{cases}$$

where the matrix $A^*_{n-1} = A^*_{n-1}(x, y_1, \ldots, y_{n-1})$ is defined by

$$A^*_{n-1} e_i = \int_{Y_n} A(e_i + \nabla_{y_n} w_i)\, dy_n.$$

It is a standard matter in homogenisation theory to check that this definition of A^*_{n-1} coincides with that given in (2.8) and that A^*_{n-1} is coercive and bounded. Furthermore, by (2.8) A^*_{n-1} has the same properties of measurability as A on $\Omega \times Y_1 \times \ldots \times Y_{n-1}$. Thus, we can iterate the previous process of eliminating and averaging the smallest scale. By induction on the number of different scales, we conclude the proof of Corollary 2.12. □

Proof of Theorem 2.14. Let us first remark that the regularity assumption on $(u_1, \ldots, u_n)$ is required in order to use

$$\sum_{k=1}^{n} [\nabla_{y_k} u_k]_\varepsilon$$

as a test function in the $(n+1)$-scale convergence definition. Such smoothness can be easily obtained if the coefficient matrix A is itself smooth in all variables $(y_1, \ldots, y_n)$.

Let us define $r_\varepsilon \in H^1(\Omega)$ by

$$r_\varepsilon(x) = u_\varepsilon(x) - u(x) - \sum_{k=1}^{n} \varepsilon_k u_k\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_k}\right).$$

We already know that r_ε converges strongly to 0 in $L^2(\Omega)$. It remains to prove the same for its gradient. To do this, we study the limit of

$$\begin{aligned}\int_\Omega [A]_\varepsilon \nabla r_\varepsilon \cdot \nabla r_\varepsilon\, dx = &\int_\Omega [A]_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon\, dx \\ &+ \int_\Omega [A]_\varepsilon \left(\nabla u(x) + \sum_{k=1}^{n} [\nabla_{y_k} u_k]_\varepsilon\right) \cdot \left(\nabla u(x) + \sum_{k=1}^{n} [\nabla_{y_k} u_k]_\varepsilon\right) dx \\ &- \int_\Omega [A]_\varepsilon \nabla u_\varepsilon \cdot \left(\nabla u(x) + \sum_{k=1}^{n} [\nabla_{y_k} u_k]_\varepsilon\right) dx \\ &- \int_\Omega [A]_\varepsilon \left(\nabla u(x) + \sum_{k=1}^{n} [\nabla_{y_k} u_k]_\varepsilon\right) \cdot \nabla u_\varepsilon\, dx + o(1),\end{aligned}$$

where $o(1)$ is a term which goes to zero as ε does. By using equation (2.3), the first term of the right-hand side of the previous equality is precisely

$$\int_\Omega f u_\varepsilon\, dx.$$

Under the smoothness assumption on $(u_1, \ldots, u_n)$, we can pass to the $(n+1)$-scale limit in the last two terms, yielding

$$\int_\Omega [A]_\varepsilon \nabla r_\varepsilon \cdot \nabla r_\varepsilon \, dx \to \int_\Omega fu \, dx$$

$$- \int_\Omega \int_{Y_1} \cdots \int_{Y_n} A\left(\nabla u + \sum_{k=1}^{n} \nabla_{y_k} u_k\right) \cdot \left(\nabla u + \sum_{k=1}^{n} \nabla_{y_k} u_k\right) dx \, dy_1 \ldots dy_n.$$

The right-hand side in the previous convergence is exactly zero by using the homogenised equation (2.6). By the coerciveness assumption (2.1), one has

$$\alpha \| \nabla r_\varepsilon \|^2_{L^2(\Omega)} \leqq \int_\Omega [A]_\varepsilon \nabla r_\varepsilon \cdot \nabla r_\varepsilon \, dx.$$

Thus, ∇r_ε converges strongly to zero in $L^2(\Omega)^N$. □

Homogenisation with an infinite number of scales

This subsection is devoted to the homogenisation of a second-order elliptic equation where the coefficient matrix has an infinite number of periodic arguments. If the matrix is not symmetric, this can be regarded as a convection–diffusion equation for a passive scalar which is convected by a fixed velocity field oscillating on an infinite number of scales. In some sense, it is a very crude (because of the assumptions of periodicity and separation of scales) model of turbulent convection.

This problem has already been addressed by Bensoussan and Lions in some unpublished notes [**6**]. We revisit their model as an illustration of our multiscale convergence method. The main point here is that we do not attack directly the infinitely many scales problem, but rather we truncate the number of scales, homogenise, and then pass to the limit as this number goes to infinity. In order to build a coefficient matrix with an infinite number of arguments, we use a limiting procedure as in [**6**].

Let $Y = [0, 1]^N$ be a unit cube of $\mathbf{R}^N$. Let α, β be two positive constants such that $0 < \alpha \leqq \beta$. Let $A^n(x, y_1, \ldots, y_n)$ be a sequence of non-necessarily symmetric matrices having an increasing number of arguments. We assume that they are continuous and Y-periodic in all the variables $(y_1, \ldots, y_n)$. Furthermore, for all $(x, y_1, \ldots, y_n) \in \Omega \times Y^n$, we have

$$\alpha |\xi|^2 \leqq A^n(x, y_1, \ldots, y_n)\xi \cdot \xi \leqq \beta |\xi|^2 \quad \forall \xi \in \mathbf{R}^N. \tag{2.11}$$

We also assume that this sequence converges to a limit matrix $A(x, y_1, \ldots)$ in the following sense:

$$\forall x \in \Omega, \quad \forall (y_k)_{k \geqq 1} \in Y^{\mathbf{N}^*}, \quad \exists A(x, y_1, \ldots) \text{ such that } \sup_{1 \leqq i, j \leqq N} |A_{ij} - A^n_{ij}| \leqq \frac{1}{n}. \tag{2.12}$$

An example of such a matrix A is given by

$$A(x, y_1, \ldots) = \sum_{n=1}^{+\infty} \theta^n B^n(x, y_1, \ldots, y_n),$$

where B^n is a sequence of matrices which satisfy (2.11) and $\theta \in]0, 1[$.

Let $(\varepsilon_k)_{k \geqq 1}$ be a sequence of scales depending on a single parameter ε, which go to zero as ε does, and assumed to be separated, i.e.

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \forall k \geqq 1. \tag{2.13}$$

By the above process, we can define an oscillating matrix $[A]_\varepsilon$ with an infinite number of arguments

$$[A]_\varepsilon(x) = A\left(x, \frac{x}{\varepsilon_1}, \ldots\right). \tag{2.14}$$

For a given source term $f \in L^2(\Omega)$, we study the homogenisation of the following equation in a bounded domain Ω:

$$\begin{cases} -\operatorname{div} [A]_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

For any value of $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ of (2.15). Furthermore, by virtue of (2.11) and (2.12), the matrix $[A]_\varepsilon$ is uniformly coercive and bounded. Thus, the sequence of solutions u_ε satisfies the *a priori* estimate

$$\| u_\varepsilon \|_{H_0^1(\Omega)} \leqq c, \tag{2.16}$$

where the constant c does not depend on ε. Up to a subsequence, u_ε converges to some limit u weakly in $H_0^1(\Omega)$. To describe the homogenised equation satisfied by u, we introduce the truncated problem

$$\begin{cases} -\operatorname{div} [A^n]_\varepsilon \nabla u_\varepsilon^n = f & \text{in } \Omega, \\ u_\varepsilon^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

The unique solution u_ε^n of (2.17) satisfies the same *a priori* estimate (2.16). Since the matrix $[A^n]_\varepsilon$ oscillates with a finite number n of scales, we can apply the results of the previous subsection (Theorem 2.11 and Corollary 2.12):

PROPOSITION 2.15. *For fixed $n \geqq 1$, the entire sequence u_ε^n converges, as ε goes to zero, to a limit u_n which is the unique solution of the truncated homogenised problem:*

$$\begin{cases} -\operatorname{div} A_n^* \nabla u_n = f & \textit{in } \Omega, \\ u_n = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.18}$$

where the matrix A_n^* is computed by the iterated homogenisation procedure

$$\begin{cases} A_n^0 = A_n(x, y_1, \ldots, y_n), \\ A_n^1 = A_n^1(x, y_1, \ldots, y_{n-1}) \textit{ homogenised matrix of } A_n^0\left(x, y_1, \ldots, y_{n-1}, \dfrac{z}{\varepsilon}\right), \\ \vdots \\ A_n^k = A_n^k(x, y_1, \ldots, y_{n-k}) \textit{ homogenised matrix of } A_n^{k-1}\left(x, y_1, \ldots, y_{n-k}, \dfrac{z}{\varepsilon}\right), \\ \vdots \\ A_n^* = A_n^n(x). \end{cases} \tag{2.19}$$

The main result of this section is the following theorem:

THEOREM 2.16. *The sequence of truncated homogenised matrices A_n^* converges uniformly in x to a limit $A^*(x)$, and more precisely there exists a constant c such that*

$$\sup_{\Omega} |A_{ij}^* - (A_n^*)_{ij}| \leqq \frac{c}{n} \quad \text{for } 1 \leqq i, j \leqq N. \tag{2.20}$$

This matrix A^* is the homogenised matrix of problem (2.15) *since the entire sequence of solutions u_ε converges weakly in $H_0^1(\Omega)$ to the unique solution u of the homogenised equation*

$$\begin{cases} -\operatorname{div} A^* \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

REMARK 2.17. For a finite number of scales, the iterated homogenised homogenisation procedure (2.19) can be understood in the following way: first homogenise the smallest scale (or the fastest variables), then iterate. However, for an infinite number of scales, there is no smallest scale, and one needs to introduce some kind of limiting procedure as we did in (2.12) and (2.20). Other choices of approximation for functions with an infinite number of arguments are possible, but we favour that presented here because of its simplicity.

Proof of Theorem 2.16. A well-known result of Boccardo and Murat [**10**] states that, if M_1^ε and M_2^ε are two matrices which satisfy

$$\alpha |\xi|^2 \leqq M_i^\varepsilon \xi \cdot \xi \leqq \beta |\xi|^2 \quad \forall \xi \in \mathbf{R}^N \text{ with } 0 < \alpha \leqq \beta,\ i = 1, 2$$

and which converges in the sense of homogenisation (H or G-convergence, see [**22**, **26**, **27**]) to two limits M_1^* and M_2^*, we have the following inequality:

$$\| M_1^* - M_2^* \|_{L^\infty(\Omega)} \leqq \frac{\beta^2}{\alpha^2} \sup_{\varepsilon > 0} \| M_1^\varepsilon - M_2^\varepsilon \|_{L^\infty(\Omega)}.$$

Applying this result to $[A^p]_\varepsilon$ and $[A^q]_\varepsilon$, we obtain

$$\| A_p^* - A_q^* \|_{L^\infty(\Omega)} \leqq \frac{\beta^2}{\alpha^2} \left(\frac{1}{p} + \frac{1}{q} \right),$$

which proves that $(A_n^*)_{n \geqq 1}$ is a Cauchy sequence in $L^\infty(\Omega)$. Thus, there exists a limit A^* which satisfies (2.20). We remark in passing that this result could have been obtained directly by inspecting the explicit formula for the homogenised matrix A_n^*.

It remains to prove that the sequence u_ε of solutions of (2.15) converges to the unique solution u of (2.21). Remark first that, since the sequence of truncated homogenised matrices A_n^* converges strongly in $L^\infty(\Omega)$ to the limit A^*, the corresponding solutions u_n converge strongly in $H_0^1(\Omega)$ to the solution u of (2.21). More precisely, we have

$$\| u_n - u \|_{H_0^1(\Omega)} \leqq \frac{c}{n}. \tag{2.22}$$

Let us estimate the difference $(u_\varepsilon^n - u_\varepsilon)$:

$$\begin{aligned}\alpha \| \nabla(u_\varepsilon^n - u_\varepsilon)\|_{L^2(\Omega)}^2 &\leqq \int_\Omega [A]_\varepsilon \nabla(u_\varepsilon^n - u_\varepsilon)\cdot\nabla(u_\varepsilon^n - u_\varepsilon)\,dx \\ &= \int_\Omega [A^n]_\varepsilon \nabla u_\varepsilon^n\cdot\nabla(u_\varepsilon^n - u_\varepsilon)\,dx \\ &\quad - \int_\Omega [A]_\varepsilon \nabla u_\varepsilon\cdot\nabla(u_\varepsilon^n - u_\varepsilon)\,dx \\ &\quad + \int_\Omega [A - A^n]_\varepsilon \nabla u_\varepsilon^n\cdot\nabla(u_\varepsilon^n - u_\varepsilon)\,dx. \end{aligned} \tag{2.23}$$

The two first terms on the right-hand side of (2.23) cancel out since they are both equal to $\int_\Omega f(u_\varepsilon^n - u_\varepsilon)\,dx$. In view of the *a priori* estimate satisfied by u_ε^n and (2.20), the last term of (2.23) is bounded by

$$\left| \int_\Omega [A - A^n]_\varepsilon \nabla u_\varepsilon^n\cdot\nabla(u_\varepsilon^n - u_\varepsilon)\,dx \right| \leqq \frac{c}{n}\, \| \nabla(u_\varepsilon^n - u_\varepsilon)\|_{L^2(\Omega)},$$

where the constant c does not depend on n or ε. Thus, (2.23) yields

$$\| u_\varepsilon^n - u_\varepsilon \|_{H_0^1(\Omega)} \leqq \frac{c}{n}. \tag{2.24}$$

Finally, from (2.22), (2.24) and Proposition 2.15, we deduce that the sequence u_ε converges weakly in $H_0^1(\Omega)$ to the limit u, solution of the homogenisation problem (2.21). □

3. Proof of Theorem 2.6

Proof of Theorem 2.6 for well-separated scales

This section provides a simple proof of Theorem 2.6 when an additional assumption is made on the n scales of oscillations. The general case, which is the focus of the next section, will be easier to understand after the following case of so-called well-separated scales.

DEFINITION 3.1. The scales $\varepsilon_1, \ldots, \varepsilon_n$ are said to be *well-separated* if and only if there exists a non-negative integer m such that

$$\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^m = 0 \quad \forall\, 1 \leqq k \leqq n-1.$$

An example of well-separated scales is given by $\varepsilon_k = \varepsilon^{\alpha_k}$, where $0 < \alpha_1 < \ldots < \alpha_n$. An example of scales which are separated in the sense of Assumption 2.1, but are not well-separated is given by $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \varepsilon |\ln \varepsilon|^{-1}$.

The following proof of Theorem 2.6 is based on H^{-1}-estimates for oscillating functions with n scales of oscillations which generalise the following classical result:

PROPOSITION 3.2. *For any continuous function $\varphi(x, y)$ defined on $\bar{\Omega} \times Y$, Y-periodic and such that*

$$\int_Y \varphi(x, y)\, dy = 0 \quad \text{for any } x \in \bar{\Omega},$$

one has the following estimate

$$\frac{1}{\varepsilon}\varphi\left(x, \frac{x}{\varepsilon}\right) \quad \text{is bounded in } H^{-1}(\Omega).$$

We generalise the result of Proposition 3.2 to the case of n well-separated scales.

THEOREM 3.3. *For any $k \in \{1, \ldots, n\}$, let E_k be the set of smooth functions $\varphi(x, y_1, \ldots, y_k)$ with compact support in Ω, periodic in $(y_1, \ldots, y_k)$ and of mean value zero with respect to the last variable y_k, i.e.*

$$E_k = \left\{\varphi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_k)] \text{ such that } \int_{Y_k} \varphi\, dy_k = 0\right\}. \tag{3.1}$$

Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are well separated (Definition 3.1). Then, for any function φ of E_k,

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \text{is bounded in } H^{-1}(\Omega).$$

COROLLARY 3.4. *Let $\varphi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]$ be a function such that for $k \in \{1, \ldots, n\}$, one has*

$$\int_{Y_k} \cdots \int_{Y_n} \varphi\, dy_k \ldots dy_n = 0. \tag{3.2}$$

Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are well separated (Definition 3.1); then

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \text{is bounded in } H^{-1}(\Omega).$$

REMARK 3.5. The assumption on the scales being well-separated is absolutely essential in Theorem 3.3. A counter-example is given in [**12**]. However, the key for the generalisation of the next section is that Theorem 3.3 holds true for a dense subset of E_k without any further assumption on the separated scales.

The convergences of Theorem 3.3 and Corollary 3.4 are stated in $H^{-1}(\Omega)$ but, since test functions φ have compact support in Ω, it holds also in the dual of $H^1(\Omega)$.

Proof of Corollary 3.4. Any function φ satisfying (3.2) can be written as a sum of functions $\varphi_j \in E_j$:

$$\varphi(x, y_1, \ldots, y_n) = \sum_{j=k}^{n} \varphi_j(x, y_1, \ldots, y_j),$$

where the functions φ_j are defined by the inductive formulae

$$\begin{cases} \varphi_n = \varphi - \int_{Y_n} \varphi \, dy_n, \\ \varphi_j = \int_{Y_{j+1}} \cdots \int_{Y_n} \varphi \, dy_{j+1} \dots dy_n - \int_{Y_j} \cdots \int_{Y_n} \varphi \, dy_j \dots dy_n \quad k \leqq j \leqq n-1. \end{cases}$$

Since each function φ_j belongs to the space E_j defined by (3.1), Theorem 2.3 implies that

$$\frac{1}{\varepsilon_j} [\varphi_j]_\varepsilon \quad \text{is bounded in } H^{-1}(\Omega)$$

and hence

$$\frac{1}{\varepsilon_k} [\varphi]_\varepsilon = \sum_{j=k}^{n} \frac{\varepsilon_j}{\varepsilon_k} \frac{1}{\varepsilon_j} [\varphi_j]_\varepsilon \quad \text{is bounded in } H^{-1}(\Omega). \quad \square$$

The proof of Theorem 3.3 requires the following lemma:

LEMMA 3.6. *For any $k \in \{1, \dots, n\}$ and any function $\varphi \in E_k$, there exists a vector-function $\Psi \in (E_k)^N$ such that*

$$\operatorname{div}_{y_k} \Psi = \varphi.$$

Furthermore, one can choose Ψ in such a way that the application S defined by:

$$E_k \to (E_k)^N,$$
$$\varphi \mapsto S\varphi = \Psi,$$

is linear and continuous with respect to the uniform continuity norm on $\Omega \times Y_1 \times \dots \times Y_k$

$$\| \varphi \| = \sup_{\substack{x \in \Omega \\ y_j \in Y_j}} |\varphi(x, y_1, \dots, y_k)|,$$

i.e. there exists a constant $c > 0$ such that

$$\| S\varphi \| \leqq c \| \varphi \|.$$

Proof. Let φ be a function of E_k. Since φ has zero mean value with respect to the variable y_k, there exists a unique function h of the problem

$$\begin{cases} \Delta_{y_k} h = \varphi & \text{in } Y_k \\ y_k \mapsto h & Y_k\text{-periodic} \end{cases} \quad \text{and} \quad \int_{Y_k} h \, dy_k = 0.$$

The operator S is then defined by

$$S\varphi = \nabla_{y_k} h.$$

By standard regularity results, one checks easily that $S\varphi$ belongs to $(E_k)^N$ and that S is continuous on E_k with respect to the uniform continuity norm. $\square$

Proof of Theorem 3.3. The proof is based on an iterative process which has already been used in [**12, 13**] for two scales. Let φ be a function of E_k. Since the average of

φ with respect to the variable y_k is zero, by virtue of Lemma 3.6, there exists $S\varphi$ in $(E_k)^N$ such that $\operatorname{div}_{y_k} S\varphi = \varphi$. Then, by taking the divergence of $[S\varphi]_\varepsilon$, one obtains

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon = \operatorname{div}[S\varphi]_\varepsilon - \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\frac{1}{\varepsilon_k}[T_\varepsilon\varphi]_\varepsilon, \tag{3.3}$$

where T_ε is the linear operator defined by

$$T_\varepsilon\varphi = \varepsilon_{k-1}\operatorname{div}_x S\varphi + \sum_{j=1}^{k-1}\frac{\varepsilon_{k-1}}{\varepsilon_j}\operatorname{div}_{y_j} S\varphi. \tag{3.4}$$

It is not difficult to check that, since the average of $S\varphi$ with respect to the variable y_k is zero, so is the average $T_\varepsilon\varphi$. Thus, $T_\varepsilon\varphi$ belongs to the set E_k.

Now, remark that in equality (3.3) $(1/\varepsilon_k)[T_\varepsilon\varphi]_\varepsilon$ is a function of the same type as $(1/\varepsilon_k)[\varphi]_\varepsilon$, but its coefficient $(\varepsilon_k/\varepsilon_{k-1})$ goes to zero, while the function $\operatorname{div}[S\varphi]_\varepsilon$ is bounded in $H^{-1}(\Omega)$. Thus, we can apply (3.3) to $T_\varepsilon\varphi$ instead of φ, and reiterating this process m times, we obtain

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon = \sum_{p=0}^{m-1}(-1)^p\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^p \operatorname{div}[S(T_\varepsilon)^p\varphi]_\varepsilon + (-1)^m\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m\frac{1}{\varepsilon_k}[(T_\varepsilon)^m\varphi]_\varepsilon. \tag{3.5}$$

since the coefficients $(\varepsilon_{k-1}/\varepsilon_j)$ are bounded in the definition (3.4) of T_ε, the function $(T_\varepsilon)^p\varphi$ is obviously bounded independently of ε in $L^\infty(\Omega)$, for any integer p.

If m has been chosen such that the scales ε_k and ε_{k-1} are separated for this value, i.e.:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m = 0,$$

then the right-hand side of (3.5) is bounded in $H^{-1}(\Omega)$, which gives the desired result. □

Before proving Theorem 2.6, we need a last lemma concerning divergence-free functions of several variables and their orthogonal.

LEMMA 3.7. *Let H be the space of 'generalised' divergence-free functions in $L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)^N]$ defined by*

$$H = \left\{\Phi \in L^2[\Omega; L^2_\#(Y_1\times\ldots\times Y_n)^N];\ \operatorname{div}_{y_n}\Phi = 0 \right.$$
$$\left.\text{and} \int_{Y_{k+1}}\ldots\int_{Y_n}\operatorname{div}_{y_k}\Phi = 0\ \forall\, 1 \leq k \leq n-1\right\}. \tag{3.6}$$

The subspace H has the following properties:

(i) *$\mathscr{D}[\Omega; C^\infty_\#(Y_1\times\ldots\times Y_n)]^N \cap H$ is dense into H.*

(ii) *The orthogonal of H is*

$$H^\perp = \left\{\sum_{k=1}^{n}\nabla_{y_k} q_k(x, y_1, \ldots, y_k) \text{ with } q_k \in L^2[\Omega\times Y_1\times\ldots\times Y_{k-1}; H^1_\#(Y_k)]\right\}. \tag{3.7}$$

REMARK 3.8. The divergence-free conditions in definition (3.6) of H have to be understood in the sense of distributions in $\mathbf{R}^N$, or equivalently, $\Phi \in H$ if and only if

$$\int_{Y_k} \cdots \int_{Y_n} \Phi \cdot \nabla_{y_k} \varphi \, dx \, dy_k \ldots dy_n = 0 \quad \text{for any } \varphi \in H^1_\#(Y_k) \quad \text{and} \quad 1 \leqq k \leqq n-1. \tag{3.8}$$

Proof of Lemma 3.7. It is easy to check that regularisation by convolution with a smooth function which is Y_k-periodic in y_k for each $k \in \{1, \ldots, n\}$, preserves condition (3.8) and thus the 'generalised' divergence-free condition. Then, by another regularisation by convolution with respect to variable x (which obviously preserves the divergence-free condition), we obtain the density result (i).

To check property (ii), we remark that $H = \cap_{k=1}^n H_k$, where

$$H_k = \left\{ \Phi \in L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)^N]; \int_{Y_{k+1}} \cdots \int_{Y_n} \operatorname{div}_{y_k} \Phi \, dy_{k+1} \ldots dy_n = 0 \right\}. \tag{3.9}$$

By a well-known result of De Rahm (see [**28**] for an elementary proof or use Fourier analysis to re-derive it in our context), the orthogonal $H_k^\perp$ is simply defined by:

$$H_k^\perp = \{\nabla_{y_k} q(x, y_1, \ldots, y_k) \text{ with } q \in L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y_k)]\}.$$

Then, the final result (3.7) can be written $H^\perp = \Sigma_{k=1}^n H_k^\perp$. From definitions (3.6) and (3.9), it remains to prove that

$$\left(\bigcap_{k=1}^n H_k \right)^\perp = \sum_{k=1}^n H_k^\perp.$$

By a classical argument, this is equivalent to checking that the subspace $\Sigma_{k=1}^n H_k$ is closed, since each subspace H_k is closed. Let us indeed show that

$$\sum_{k=1}^n H_k = L^2(\Omega \times Y_1 \times \ldots \times Y_n)^N.$$

Let $\Phi \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)^N$; one has

$$\Phi = \int_{Y_n} \Phi \, dy_n + \left(\Phi - \int_{Y_n} \Phi \, dy_n \right),$$

where the first term on the right-hand side belongs to H_n (because it does not depend on y_n) and the second term to H_{n-1}. Thus, $H_n + H_{n-1} = L^2(\Omega \times Y_1 \times \ldots \times Y_n)^N$, which concludes the proof. □

Proof of Theorem 2.6. Since the function u_ε is bounded in $H^1(\Omega)$, by application of Theorem 2.4, there exist two limits $u_0(x, y_1, \ldots, y_n) \in L^2(\Omega; \times Y_1 \times \ldots \times Y_n)$ and $\xi_0(x, y_1, \ldots, y_n) \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)^N$, such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{(n+1)\text{-scale}} u_0,$$

$$\nabla u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \xi_0.$$

Let us prove in a first step that u_0 does not depend on the variables $y_1, \ldots, y_n$.

First step. Let $\Phi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N$. By integration by parts, one has

$$\varepsilon_n \int_\Omega \nabla u_\varepsilon \cdot [\Phi]_\varepsilon \, dx = -\varepsilon_n \int_\Omega u_\varepsilon [\operatorname{div}_x \Phi]_\varepsilon \, dx - \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \int_\Omega u_\varepsilon [\operatorname{div}_{y_k} \Phi]_\varepsilon \, dx - \int_\Omega u_\varepsilon [\operatorname{div}_{y_n} \Phi]_\varepsilon \, dx.$$

Then passing to the $(n+1)$-scale limit in the previous equality yields

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} u_0 \operatorname{div}_{y_n} \Phi \, dx \, dy_1 \ldots dy_n = 0.$$

Thus, u_ε does not depend on y_n. Now, we choose a test function Φ which does not depend on y_n too. Repeating the same argument, we obtain that u_0 does not depend on y_{n-1}. By induction, we deduce that u_0 depends only on x.

Second step. Let us characterise $\xi_0(x, y_1, \ldots, y_n)$ the $(n+1)$-scale limit of ∇u_ε. Let $\Phi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N$ be a smooth function satisfying the 'generalised' divergence-free condition (3.6). By integration by parts, one has

$$\int_\Omega \nabla u_\varepsilon [\Phi]_\varepsilon \, dx = -\int_\Omega u_\varepsilon [\operatorname{div}_x \Phi]_\varepsilon \, dx - \sum_{k=1}^{n-1} \frac{1}{\varepsilon_k} \int_\Omega u_\varepsilon [\operatorname{div}_{y_k} \Phi]_\varepsilon \, dx, \tag{3.10}$$

since $\operatorname{div}_{y_n} \Phi = 0$. The function $\operatorname{div}_{y_k} \Phi$ satisfies the assumptions of Corollary 3.4, i.e.

$$\int_{Y_{k+1}} \ldots \int_{Y_n} \operatorname{div}_{y_k} \Phi \, dy_{k+1} \ldots dy_n = 0,$$

thus

$$\frac{1}{\varepsilon_{k+1}} [\operatorname{div}_{y_k} \Phi]_\varepsilon$$

is bounded in the dual of $H^1(\Omega)$. Since

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0, \quad \frac{1}{\varepsilon_k} [\operatorname{div}_{y_k} \Phi]_\varepsilon$$

converges strongly to zero in the dual of $H^1(\Omega)$ for $1 \leqq k \leqq n-1$. By using the previous strong convergence and by passing to the $(n+1)$-scale limit in (3.10), one obtains

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} \xi_0 \cdot \Phi \, dy_1 \ldots dy_n = -\int_\Omega \int_{Y_1} \ldots \int_{Y_n} u_0 \operatorname{div}_x \Phi \, dx \, dy_1 \ldots dy_n,$$

which implies that $u_0 \in H^1(\Omega)$. Another integration by parts on the right-hand side with respect to x yields

$$\int_\Omega \int_{Y_1} \ldots \int_{Y_n} (\xi_0 - \nabla u_0) \cdot \Phi \, dy_1 \ldots dy_n = 0$$

$$\text{for any } \Phi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H.$$

By the density result (i) of Lemma 3.6, this is true for any $\Phi \in H$, thus $\xi_0 - \nabla u_0 \in H^\perp$.

In view of the definition (3.7) of $H^{\perp}$, this proves that

$$\nabla u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \xi_0 = \nabla u_0(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k),$$

with $u_k(x, y_1, \ldots, y_k) \in L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y_k)]^N$ for $1 \leqq k \leqq n$.

To conclude the proof of Theorem 2.6, it remains to check that any $(n+1)$-scale limit $(u, u_1, \ldots, u_n)$ is attained. For smooth periodic functions $(u_1, \ldots, u_n)$, this is obvious by simply taking

$$u_\varepsilon(x) = u(x) + \sum_{k=1}^{n} \varepsilon_k u_k\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_k}\right).$$

Since smooth functions are dense in $L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y_k)]^N$, a standard diagonalisation argument shows that the same is true for any $(n+1)$-scale limit (see [**2**, Lemma 1.3] for details in the two-scale case). □

Proof of Theorem 2.7. Let ξ_ε be a bounded sequence in $L^2(\Omega)^N$ such that

$$\operatorname{div} \xi_\varepsilon = 0 \quad \text{in } \Omega$$

and

$$\xi_\varepsilon \xrightarrow{(n+1)\text{-scale}} \xi_0(x, y_1, \ldots, y_n).$$

Multiplying the divergence-free equation by $\varepsilon_k[\varphi]_\varepsilon$, where $\varphi(x, y_1, \ldots, y_k)$ is a smooth periodic function with compact support in Ω, yields

$$\lim_{\varepsilon \to 0} \int_\Omega \xi_\varepsilon \cdot \nabla(\varepsilon_k[\varphi]_\varepsilon)\, dx = \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \xi_0 \cdot \nabla_{y_k} \varphi \, dy_1 \ldots dy_n = 0.$$

Since the choice of φ is arbitrary, another integration by parts gives

$$\int_{Y_{k+1}} \cdots \int_{Y_n} \operatorname{div}_{y_k} \xi_0 \, dy_{k+1} \ldots dy_n = 0 \quad \text{for any } 1 \leqq k \leqq n-1.$$

Similarly, by taking a test function $\varphi(x)$, we get

$$\int_{Y_1} \cdots \int_{Y_n} \operatorname{div}_x \xi_0 \, dy_1 \ldots dy_n = 0.$$

To complete the proof of Theorem 2.7, it remains to prove that any function $\xi_0 \in L^2[\Omega; L^2_\#(Y_1 \times \ldots \times Y_n)]^N$ satisfying the 'generalised' divergence-free condition is attained as an $(n+1)$-scale limit of a divergence-free sequence. Let us assume for a moment that ξ_0 is a smooth function. Obviously, the sequence defined by

$$\xi_\varepsilon(x) = [\xi_0]_\varepsilon(x) = \xi_0\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right)$$

$(n+1)$-scale converges to ξ_0. However, ξ_ε is not divergence-free in Ω.

To remedy this inconvenience, we recall that $L^2(\Omega)^N$ is the direct sum of divergence-free functions and gradients of $H^1_0(\Omega)$ functions. Thus, there exists a unique decomposition

$$\xi_\varepsilon = v_\varepsilon + \nabla p_\varepsilon,$$

where v_ε is a divergence-free, bounded sequence in $L^2(\Omega)^N$, and p_ε is bounded in $H_0^1(\Omega)$. The sequence v_ε $(n+1)$-scale converges to a limit v_0 which obviously satisfies the 'generalised' divergence-free condition. By virtue of Theorem 2.6, the sequence ∇p_ε $(n+1)$-scale converges to a 'generalised' gradient limit which is precisely orthogonal to all 'generalised' divergence-free fields. Thus, we conclude that v_0 is equal to ξ_0 which is therefore attained by the divergence-free sequence v_ε. In the case where ξ_0 is not a smooth function, a standard diagonalisation argument achieves the proof (see [**2**, Lemma 1.13] for details in the two-scale case). □

Proof of Theorem 2.6 in the general case

The aim of this subsection is to generalise the results of the first subsection in the case where the scales are simply separated (Assumption 2.1) and not necessarily well separated (Definition 3.1). The key ingredient for the proof of Theorem 2.6 in the case of well-separated scales is the H^{-1}-estimate of oscillating sequences provided by Theorem 3.3. As already seen in Remark 3.5, such an estimate does not hold true in general for simply separated scales. However, the following result extends Theorem 3.3 for a strictly smaller class of test functions.

THEOREM 3.9. *Let F_k be the set of smooth functions $\varphi(x, y_1, \ldots, y_k)$ with compact support in Ω, periodic in $(y_1, \ldots, y_k)$, of mean value zero with respect to the last variable y_k, and satisfying a growth condition with respect to the variables $(y_1, \ldots, y_{k-1})$, i.e.*

$$F_k = \left\{ \varphi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_k)]; \left[\begin{array}{l} \displaystyle\int_{Y_k} \varphi \, dy_k = 0, \\ \exists\, \delta > 0,\ \forall\, p \in \mathbf{N}^* \, \|\nabla^p \varphi\| + \|\nabla^p \nabla_x \varphi\| \leqq \delta^p \end{array} \right. \right\}, \tag{3.11}$$

where ∇^p denotes all the derivatives until the p*th order with respect to the variables $(y_1, \ldots, y_{k-1})$. and $\|\cdot\|$ is the uniform continuity norm, i.e.:*

$$\|\varphi\| = \sup_{\substack{x \in \Omega \\ y_j \in Y_j}} |\varphi(x, y_1, \ldots, y_k)|.$$

Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are simply separated (Assumption 2.1). Then, for any function φ of F_k,

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \text{is bounded in } H^{-1}(\Omega).$$

COROLLARY 3.10. *Let $\varphi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]$ be a function such that, for a given index $k \in \{1, \ldots, n\}$,*

$$\int_{Y_k} \cdots \int_{Y_n} \varphi \, dy_k \ldots dy_n = 0 \quad \text{and} \quad \exists\, \delta > 0,\ \forall\, p \in \mathbf{N}^* \, \|\nabla^p \varphi\| + \|\nabla^p \nabla_x \varphi\| \leqq \delta^p,$$

where ∇^p denotes all the derivatives until the p*th order with respect to the variables $(y_1, y_2, \ldots, y_{n-1})$. Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are separated (Assumption 2.1);*

then

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \textit{is bounded in } H^{-1}(\Omega).$$

Proof. The proof of Corollary 3.10 is a simple adaptation of that of Corollary 3.4, so we omit it. □

The class of functions which satisfy the growth condition introduced in Theorem 3.9 and Corollary 2.10 is far from being empty. Indeed, all functions which are partial sums of Fourier series with respect to the variables $(y_1, y_2, \ldots, y_n)$ satisfy such a growth condition. Furthermore, they form a dense subset of the space H defined by (3.6), as stated in the next lemma.

LEMMA 3.11. *Let us again consider the space H of 'generalised' divergence-free functions, defined in Lemma 3.7. Let K be the subspace of H composed of smooth functions $\Phi(x, y_1, \ldots, y_n)$ satisfying a growth condition with respect to the variables $(y_1, y_2, \ldots, y_{n-1})$, i.e.*

$$K = \{\Phi \in \mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H;\ \exists\, \delta > 0,\ \forall\, p \in \mathbf{N}^* \, \|\nabla^p \Phi\| + \|\nabla^p \nabla_x \Phi\| \leqq \delta^p\}, \tag{3.12}$$

where ∇^p denotes all the derivatives until the p*th order with respect to variables $(y_1, y_2, \ldots, y_{n-1})$. Then, K is dense into H.*

Proof. By Lemma 3.7, the space $\mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H$ is dense into the space H. Furthermore, the subspace of $\mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H$ composed of partial sums of Fourier's series with respect to the variables $(y_1, y_2, \ldots, y_n)$, is dense into $\mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H$, since the convolution by the Dirichlet's kernel preserves the 'generalised' divergence-free condition. Finally, it is easy to see that the partial sums of Fourier's series with respect to the variables $(y_1, y_2, \ldots, y_n)$ and which belong to $\mathscr{D}[\Omega; C^\infty_\#(Y_1 \times \ldots \times Y_n)]^N \cap H$, satisfy the growth condition (3.13), which concludes the proof. □

Proof of Theorem 2.6. If we assume for a moment that Theorem 3.9 holds true, the proof of Theorem 2.6 in the general case of separated scales is simply a repetition of that in the case of well-separated scales (see the first subsection). The only change is that the test functions are chosen in the space K defined above in Lemma 3.11. Since K is dense in H, this does not change the conclusion of Theorem 2.6. □

Finally, it remains to prove Theorem 3.9. We are going to adapt the ideas of Theorem 3.3 for the special test functions satisfying the growth condition (3.11). We begin with a lemma which generalises Lemma 3.6 to the present setting.

LEMMA 3.12. *Let $k \in \{1, \ldots, n\}$. Let S be the linear application from E_k to $(E_k)^N$ defined in Lemma 3.6, and such that, for any $\varphi \in E_k$, one has*

$$\operatorname{div}_{y_k} S\varphi = \varphi.$$

Then, for any value of $(a_1, \ldots, a_{k-1})$ in $[0, 1]^{k-1}$, a linear operator T from F_k into F_k (see (3.11)) is defined by

$$T\varphi = \sum_{j=1}^{k-1} a_j \operatorname{div}_{y_j} S\varphi, \tag{3.13}$$

which satisfies the growth condition

$$\| T^p\varphi \| \leqq c^p \| \nabla^p \varphi \| \quad \textit{and} \quad \| \operatorname{div}_x (ST^p\varphi) \| \leqq c^{p+1} \| \nabla^p \nabla_x \varphi \| \quad \forall \varphi \in F_k, \quad \forall p \in \mathbf{N}, \tag{3.14}$$

where ∇^p *denotes all the derivatives until the* p*th order with respect to variables* $(y_1, y_2, \ldots, y_{k-1})$, *and the norm* $\|\cdot\|$ *and the constant* c *(independent of the values* $a_1, \ldots, a_{k-1}$*) are the same as that of Lemma* 3.6.

Proof. Let $\varphi \in F_k$ and $\Psi = S\varphi \in (E_k)^N$. The proof is divided into five steps.

First step. Let ∂^p be a partial derivative of pth order with respect to the variables $(y_1, y_2, \ldots, y_{k-1})$, and ∂_{x_i} be a partial derivative of first order with respect to the variable x. Let us prove that

$$\partial^p \circ S = S \circ \partial^p, \quad \partial_{x_i} \circ S = S \circ \partial_{x_i}, \quad \partial_{x_i} \circ T = T \circ \partial_{x_i}. \tag{3.15}$$

From the definition of operator S in the proof of Lemma 3.6, one has

$$\Psi = S\varphi = \nabla_{y_k} h \text{ where } \Delta_{y_k} h = \varphi, \quad \text{then } \partial^p \Psi = \nabla_{y_k}(\partial^p h) \text{ where } \Delta_{y_k}(\partial^p h) = \partial^p \varphi.$$

Thus $\partial^p(S\varphi) = S(\partial^p\varphi)$, which proves the first commutation in (3.15). The proofs of the other ones are similar.

Second step. Let us prove that

$$\| T\varphi \| \leqq c \| \nabla \varphi \|. \tag{3.16}$$

The following upper bound holds:

$$\begin{aligned} \| T\varphi \| &\leqq \sum_{j=1}^{k-1} a_j \| \operatorname{div}_{y_j} \Psi \| \quad \text{where } \Psi = S\varphi \\ &\leqq \sum_{j=1}^{k-1} \| \operatorname{div}_{y_j} \Psi \| \leqq \sum_{j=1}^{k-1} \sum_{i=1}^{N} \left\| \frac{\partial \Psi_i}{\partial y_{ji}} \right\| \quad \text{since } 0 \leqq a_j \leqq 1 \\ &\leqq \sum_{j=1}^{k-1} \sum_{i=1}^{N} \left\| S\left(\frac{\partial \varphi}{\partial y_{ji}}\right) \right\| \quad \text{from (3.15)} \\ &\leqq \sum_{j=1}^{k-1} \sum_{i=1}^{N} c \left\| \frac{\partial \varphi}{\partial y_{ji}} \right\| = c \| \nabla \varphi \| \quad \text{by virtue of Lemma 3.6,} \end{aligned}$$

which concludes the second step.

Third step. Let us prove that

$$\| \nabla^p(T\varphi) \| \leqq c \| \nabla^{p+1} \varphi \| \quad \forall p \in \mathbf{N}. \tag{3.17}$$

Let ∂^p be a partial derivative of pth order with respect to the variables $(y_1, y_2, \ldots, y_{k-1})$:

$$\begin{aligned} \partial^p(T\varphi) &= \sum_{j=1}^{k-1} a_j \operatorname{div}_{y_j} (\partial^p \Psi) \quad \text{where } \Psi = S\varphi \\ &= \sum_{j=1}^{k-1} a_j \operatorname{div}_{y_j} S(\partial^p \varphi) = T(\partial^p \varphi) \quad \text{from (3.15).} \end{aligned}$$

From the previous equality and (3.16), one deduces

$$\| \partial^p(T\varphi) \| \leqq c \| \nabla(\partial^p\varphi) \| .$$

By adding the previous equalities for all the partial derivatives ∂^p, one thus obtains (3.17), where the constant c is precisely the same as that in (3.16).

Fourth step. Let us prove by induction that

$$\| T^p\varphi \| \leqq c^p \| \nabla^p\varphi \| \quad \forall p \in \mathbf{N}. \tag{3.18}$$

Estimate (3.18) is true for p equal to 1 owing to (3.16). Assume that it is true for $p-1$; then:

$$\| T^p\varphi \| = \| T^{p-1}(T\varphi) \| \leqq c^{p-1} \| \nabla^{p-1}(T\varphi) \| \leqq c^p \| \nabla^p\varphi \| \quad \text{by virtue of (3.17).}$$

Thus estimate (3.18) is also true for p.

Fifth step. Let us prove that

$$\| \operatorname{div}_x (ST^p\varphi) \| \leqq c^{p+1} \| \nabla^p\nabla_x\varphi \| \quad \forall p \in \mathbf{N}. \tag{3.19}$$

One has

$$\begin{aligned} \| \operatorname{div}_x (ST^p\varphi) \| &\leqq \sum_{i=1}^{N} \| \partial_{x_i}(ST^p\varphi) \| = \sum_{i=1}^{N} \| ST^p(\partial_{x_i}\varphi) \| \quad \text{from (3.15)} \\ &\leqq c \sum_{i=1}^{N} \| T^p(\partial_{x_i}\varphi) \| \quad \text{by the properties of } S \\ &\leqq c^{p+1} \sum_{i=1}^{N} \| \nabla^p(\partial_{x_i}\varphi) \| \quad \text{from (3.18)} \\ &= c^{p+1} \| \nabla^p\nabla_x\varphi \| . \end{aligned}$$

This proves (3.19) which, combined with (3.18), is nothing but (3.14). Finally, as a direct consequence of (3.15), the function $T\varphi$ satisfies the growth condition (3.11) which proves that T maps F_k into F_k. □

Proof of Theorem 3.9. Let φ be a function of F_k defined by (3.11). By definition, there exists $\delta > 0$ such that

$$\| \nabla^p\varphi \| + \| \nabla^p\nabla_x\varphi \| \leqq \delta^p \quad \forall p \in \mathbf{N}^*. \tag{3.20}$$

Let c be the constant in Lemma 3.6 and 3.12. For sufficiently small ε, from Assumption 2.1, we deduce

$$\frac{\varepsilon_{k-1}}{\varepsilon_j} \leqq 1 \quad \forall 1 \leqq j \leqq k-1 \quad \text{and} \quad c\delta \frac{\varepsilon_k}{\varepsilon_{k-1}} = r_\varepsilon \leqq \frac{1}{2}. \tag{3.21}$$

By taking the divergence of $[S\varphi]_\varepsilon$, one has

$$\frac{1}{\varepsilon_k} [\varphi]_\varepsilon = \operatorname{div} [S\varphi]_\varepsilon - [\operatorname{div}_x S\varphi]_\varepsilon - \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right) \frac{1}{\varepsilon_k} [T_\varepsilon\varphi]_\varepsilon, \tag{3.22}$$

where T_ε is the linear operator defined by

$$T_\varepsilon\varphi = \sum_{j=1}^{k-1} \frac{\varepsilon_{k-1}}{\varepsilon_j} \operatorname{div}_{y_j} S\varphi.$$

In passing, we emphasise the differences between the above definition of T_ε and (3.4), which was introduced in the first subsection for well-separated scales. Here, the operator T_ε does not include any term of the type $\operatorname{div}_x S\varphi$, because we do not control any growth condition on iterated derivatives with respect to the variable x. Then, as in the proof of Theorem 3.3, iterating (3.22) m times yields

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon = \sum_{p=0}^{m-1} (-1)^p \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^p (\operatorname{div}[S(T_\varepsilon)^p\varphi]_\varepsilon - [\operatorname{div}_x (S(T_\varepsilon)^p\varphi)]_\varepsilon) + (-1)^m \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m \frac{1}{\varepsilon_k}[(T_\varepsilon)^m\varphi]_\varepsilon. \tag{3.23}$$

To estimate the norm of (3.23) in $H^{-1}(\Omega)$, we remark that, since Ω is bounded, for any smooth vector-function $\Psi(x, y_1, \ldots, y_k)$ periodic in $(y_1, \ldots, y_k)$, one has

$$\|[\Psi]_\varepsilon\|_{L^2(\Omega)} + \|\operatorname{div}[\Psi]_\varepsilon\|_{H^{-1}(\Omega)} \leqq C(\Omega)\|\Psi\|, \tag{3.24}$$

where $C(\Omega)$ is a constant which only depends on Ω and $\|\cdot\|$ the usual norm in $L^\infty(\Omega \times Y_1 \times \ldots \times Y_n)$. Thus, by integrating by parts the terms $\operatorname{div}[S(T_\varepsilon)^p\varphi]_\varepsilon$, we deduce from (3.23) that

$$\frac{1}{\varepsilon_k}\|[\varphi]_\varepsilon\|_{H^{-1}(\Omega)} \leqq C(\Omega)\left[\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m \|(T_\varepsilon)^m\varphi\| + \sum_{p=0}^{m-1}\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^p (\|S(T_\varepsilon)^p\varphi\| + \|\operatorname{div}_x[S(T_\varepsilon)^p\varphi]\|)\right];$$

combined with (3.14), it yields

$$\frac{1}{\varepsilon_k}\|[\varphi]_\varepsilon\|_{H^{-1}(\Omega)} \leqq C(\Omega)\left[\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m c^m\|\nabla^m\varphi\| + \sum_{p=0}^{m-1}\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^p c^{p+1}(\|\nabla^p\varphi\| + \|\nabla^p\nabla_x\varphi\|)\right].$$

By using (3.20) and (3.21), we finally obtain

$$\begin{aligned}\frac{1}{\varepsilon_k}\|[\varphi]_\varepsilon\|_{H^{-1}(\Omega)} &\leqq C(\Omega)\left[\frac{1}{\varepsilon_k}(r_\varepsilon)^m + c(\|\varphi\| + \|\nabla_x\varphi\|) + \sum_{p=1}^{m-1} c(r_\varepsilon)^p\right]\\ &\leqq C(\Omega)\left[\frac{1}{\varepsilon_k}(r_\varepsilon)^m + c(\|\varphi\| + \|\nabla_x\varphi\|) + \frac{cr_\varepsilon}{1-r_\varepsilon}\right].\end{aligned} \tag{3.25}$$

For fixed ε, we let m go to infinity in (3.25). Since $r_\varepsilon \leqq \frac{1}{2}$, this proves that

$$\frac{1}{\varepsilon_k}\|[\varphi]_\varepsilon\|_{H^{-1}(\Omega)}$$

is bounded in $H^{-1}(\Omega)$ independently of ε, which concludes the proof. □

Finally, we remark that letting ε go to 0 implies that r_ε goes to zero too, which yields the following estimate:

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon_k} \| [\varphi]_\varepsilon \|_{H^{-1}(\Omega)} \leqq cC(\Omega)(\|\varphi\| + \|\nabla_x \varphi\|),$$

where c is the constant defined in Lemma 3.6 and 3.12, and $C(\Omega)$ the constant defined in (3.24).

4. Multiscale convergence for perforated domains

This section is devoted to the generalisation of the results of Sections 2 and 3 to the case of periodically perforated domains. Our main result is a homogenisation theorem for the Neumann problem in a multiscale, periodically perforated domain. This result is obtained by application of the multiscale method, and we emphasise that no extension operators are required and that there is no restrictive geometrical assumption on the perforations. Homogenisation of the Neumann problem in simply periodically perforated domains has many applications and has been widely studied (see [**1–3**, **15**], and the references therein). We believe the present paper is the first to address the Neumann problem in domains perforated on multiple scales, although we know of some published work for the Dirichlet problem in perforated domains with double periodicity (see [**19**]), and the ongoing work of Mekkaoui and Picard.

Main results

We keep the notation of the previous sections and we add the following: for each scale $k \in \{1, \ldots, n\}$, the unit cell Y_k is divided in a material part Y_k^* and a hole T_k: Y_k^* is an open subset of Y_k, and $T_k = Y_k \backslash \overline{Y_k^*}$. We denote by $\chi_k(y_k)$ the Y_k-periodic characteristic function of the material part Y_k^*, extended by Y_k-periodicity to $\mathbf{R}^N$. We denote by $E_\#(Y_k^*)$ the open set obtained by periodic repetition of Y_k^*, i.e.

$$E_\#(Y_k^*) = \{ y \in \mathbf{R}^N;\ \chi_k(y) = 1 \}.$$

Throughout this section, we assume that the material parts are connected:

Assumption 4.1. $E_\#(Y_k^*)$ is a connected open set of $\mathbf{R}^N$, with a Lipschitz boundary, for all $k \in \{1, \ldots, n\}$. (The holes $E_\#(T_k)$ can be either connected or not.)

As usual, we denote by $H_\#^1(Y_k^*)$ the set of functions $\varphi(y_k)$ in $H_{\mathrm{loc}}^1[E_\#(Y_k^*)]$ which are Y_k-periodic. Then, we denote by $\mathscr{D}_\#(Y_k^*)$ the set of periodic smooth functions $\varphi(y_k)$ in $C_\#^\infty(Y_k)$, such that, extended by Y_k-periodicity to $\mathbf{R}^N$, their support is contained in $E_\#(Y_k^*)$ (in other words, these functions have compact support in the image of Y_k^* in the torus). We are now in a position to define our multiscale perforated domain.

Definition 4.2. Let Ω be a bounded set of $\mathbf{R}^N$, with a Lipschitz boundary. From the fixed domain Ω, we define a multiscale perforated domain Ω_ε by

$$\Omega_\varepsilon = \Omega \cap \left(\bigcap_{k=1}^{n} E_\#(\varepsilon_k Y_k^*) \right).$$

In other words, we have

$$\Omega_\varepsilon = \{ x \in \Omega;\ [\chi]_\varepsilon(x) = 1 \},$$

where $\chi(y_1, \ldots, y_n)$ is the multiscale characteristic function

$$\prod_{k=1}^{n} \chi_k(y_k).$$

In this perforated domain Ω_ε, we consider the following Neumann problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.1}$$

which has a unique solution in $H^1(\Omega_\varepsilon)$, for a given source term $f \in L^2(\Omega)$. From equation (4.1), we easily obtain an *a priori* estimate

$$\| u_\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqq c, \tag{4.2}$$

where the constant c does not depend on ε. The homogenisation problem for (4.1) is to investigate in which sense the sequence u_ε converges to a limit u, and what is the homogenised equation satisfied by u.

REMARK 4.3. The Neumann problem (4.1) has many variants. For example, we could have replaced, with no further difficulties, the Laplacian by an oscillating second-order elliptic operator as in Section 2. It is also possible to replace the single Neumann condition on $\partial\Omega_\varepsilon$, by a Dirichlet condition on $\partial\Omega \cap \partial\Omega_\varepsilon$ and a Neumann one on $\partial\Omega_\varepsilon \backslash \partial\Omega$ (see [**3**] for details in the two-scale case). For simplicity, we focus on the simple problem (4.1), which contains a zero-order term, in order to easily derive the *a priori* estimate (4.2).

A classical difficulty with homogenisation in perforated domains is that the sequence of solutions u_ε is not bounded in a fixed Sobolev space independent of ε. Consequently, we cannot extract a converging subsequence for some usual weak topology. One way to circumvent this problem is to define a bounded extension operator from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$ (see [**1, 15**]). Another approach is given by the multiscale convergence method, which allows us to extract converging subsequences (in the sense of $(n+1)$-scale convergence) from the *a priori* estimate (4.2) in $H^1(\Omega_\varepsilon)$, without using any extension techniques (apart from the trivial extension by zero in the holes $\Omega \backslash \Omega_\varepsilon$).

The main result of this section is the following theorem:

THEOREM 4.4. *Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are well separated (see Definition 3.1). Denote by $\tilde{}$ the extension by zero in the holes $\Omega \backslash \Omega_\varepsilon$. The sequences $\tilde{u}_\varepsilon$ and $\widetilde{\nabla u_\varepsilon}$ $(n+1)$-scale converge, respectively, to*

$$\chi(y_1, \ldots, y_n)u(x) \quad \textit{and to} \quad \chi(y_1, \ldots, y_n)\left[\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k)\right],$$

where $(u, u_1, \ldots, u_n)$ is the unique solution in

$$V^* = H^1(\Omega) \times \prod_{k=1}^{n} L^2[\Omega \times Y_1^* \times \ldots \times Y_{k-1}^*; H_\#^1(Y_k^*)/\mathbb{R}] \tag{4.3}$$

of the $(n+1)$*-scale homogenised system*

$$
\begin{cases}
-\operatorname{div}_{y_n}\left(\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)=0 \quad \text{in } Y_n^*,\\
\left(\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)\cdot\nu=0 \quad \text{on } \partial T_n,\\
-\operatorname{div}_{y_k}\left[\int_{Y_{k+1}^*}\cdots\int_{Y_n^*}\left(\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)dy_{k+1}\dots dy_n\right]=0 \quad \text{in } Y_k^*\\
\qquad 1\leqq k\leqq n-1,\\
\left[\int_{Y_{k+1}^*}\cdots\int_{Y_n^*}\left(\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)dy_{k+1}\dots dy_n\right]\cdot\nu=0 \quad \text{on } \partial T_k,\\
\left[\int_{Y_1^*}\cdots\int_{Y_n^*}\left[-\operatorname{div}_x\left(\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j\right)+u(x)-f(x)\right]dy_1\dots dy_n=0 \quad \text{in } \Omega.
\end{cases}
\tag{4.4}
$$

Denoting by θ_k the material volume fraction at the kth scale (i.e. $\theta_k=|Y_k^*|$), we can eliminate the microscopic variables to get the usual homogenised equation.

COROLLARY 4.5. *The* $(n+1)$*-scale limit* $u(x)$ *of the sequence* u_ε *is also the unique solution in* $H^1(\Omega)$ *of the homogenised problem*

$$
\begin{cases}
-\operatorname{div}(A^*\nabla u)+\theta u=\theta f & \text{in } \Omega,\\
(A^*\nabla u)\cdot n=0 & \text{on } \partial\Omega,
\end{cases}
\tag{4.5}
$$

where $\theta=\Pi_{k=1}^n\theta_k$ *is the overall volume fraction of material, and* A^* *is the homogenised matrix defined by the inductive formulae*

$$
\begin{cases}
A_n^*=Id,\\
A_k^*=\text{periodic homogenisation of } A_{k+1}^* \quad \text{in } Y_{k+1}^* \quad 1\leqq k\leqq n-1,\\
A^*=\text{periodic homogenisation of } A_1^* \quad \text{in } Y_1^*.
\end{cases}
\tag{4.6}
$$

In other words, we have, for any $\xi\in\mathbf{R}^N$,

$$
A_k^*\xi=\int_{Y_{k+1}^*}A_{k+1}^*(\xi+\nabla_{y_{k+1}}w_{k+1}^\xi)\,dy_{k+1},
$$

with w_{k+1}^ξ *the unique solution in* $H_\#^1(Y_{k+1}^*)/\mathbf{R}$ *of*

$$
\begin{cases}
-\operatorname{div}_{y_{k+1}}[A_{k+1}^*(\xi+\nabla_{y_{k+1}}w_{k+1}^\xi)]=0 & \text{in } Y_{k+1}^*,\\
[A_{k+1}^*(\xi+\nabla_{y_{k+1}}w_{k+1}^\xi)]\cdot\nu=0 & \text{on } \partial T_{k+1}.
\end{cases}
$$

The key ingredient for proving Theorem 4.4 and Corollary 4.5 is the following generalisation of Theorem 2.6 about $(n+1)$-scale convergence of gradients in perforated domains:

THEOREM 4.6. *Assume that the scales* $\varepsilon_1,\dots,\varepsilon_n$ *are well separated (see Definition* 3.1). *Then, for any sequence* u_ε *bounded in* $H^1(\Omega_\varepsilon)$, *namely*

$$
\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqq c,
$$

where c is a constant independent of ε, *there exists* $u(x) \in H^1(\Omega)$ *and n functions* $u_k(x, y_1, \ldots, y_k)$ *in* $L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_\#(Y^*_k)]$ *such that, up to a subsequence, we have*

$$\tilde{u}_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n)u(x),$$

$$\tilde{\nabla} u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n)\left[\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k)\right].$$

REMARK 4.7. The proof of Theorem 4.6 (see the second subsection) is not a straightforward generalisation of that of Theorem 2.6. Of course, if we assume the existence of a bounded operator from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$ then Theorem 4.6 is a direct consequence of the previous results of Sections 2 and 3. In the present situation, it seems quite intricate to build such an extension operator. Thus, the main interest of Theorem 4.6 is that it does not require any extension operator. Remark also that this difficulty of working in perforated domains forces us to assume that the scales are well separated in the sense of Definition 3.1. Our method of proof does not work in the case of simply separated scales, because we use test functions which vanish on the holes boundaries, and thus cannot satisfy a growth condition as (3.14).

Proof of Theorem 4.4. The proof is very similar to that of Theorem 2.11. Thanks to the *a priori* estimate (4.2) on the sequence u_ε, we can apply Theorem 4.6. There exists $(u, u_1, \ldots, u_n) \in V^*$ (defined by (4.3)) such that, up to a subsequence,

$$\tilde{u}_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n)u(x),$$

$$\tilde{\nabla} u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n)\left[\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k)\right].$$

Let us fix a test function

$$(\varphi, \varphi_1, \ldots, \varphi_n) \in C^\infty(\bar{\Omega}) \times \prod_{k=1}^{n} C^\infty[\bar{\Omega}; C^\infty_\#(Y_1 \times \ldots \times Y_k)].$$

Multiplying equation (4.1) by

$$\varphi + \sum_{k=1}^{n} \varepsilon_k [\varphi_k]_\varepsilon,$$

taking into account the Neumann boundary condition, we obtain

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \left(\nabla_x \varphi + \sum_{k=1}^{n} [\nabla_{y_k} \varphi_k]_\varepsilon\right) dx + \int_{\Omega_\varepsilon} u_\varepsilon \varphi \, dx = \int_{\Omega_\varepsilon} f\varphi \, dx + o(1), \tag{4.7}$$

where $o(1)$ denotes a term which goes to zero as ε does. By introducing the characteristic function $\chi(y_1, \ldots, y_n)$ and the extension $\tilde{}$ by zero in the holes $\Omega \backslash \Omega_\varepsilon$, we can replace Ω_ε by Ω in the integrals of (4.7):

$$\int_{\Omega} \tilde{\nabla} u_\varepsilon \cdot \left(\nabla_x \varphi + \sum_{k=1}^{n} [\nabla_{y_k} \varphi_k]_\varepsilon\right) dx + \int_{\Omega} \tilde{u}_\varepsilon \varphi \, dx = \int_{\Omega} [\chi]_\varepsilon f\varphi \, dx + o(1).$$

Then, passing to the $(n+1)$-scale limit yields

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} \left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right) \cdot \left(\nabla_x \varphi + \sum_{k=1}^n \nabla_{y_k} \varphi_k\right) dx\, dy_1 \ldots dy_n$$

$$+ \int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} u\varphi\, dx\, dy_1 \ldots dy_n = \int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} f\varphi\, dx\, dy_1 \ldots dy_n.$$

By density, this equality holds true for any test function $(\varphi, \varphi_1, \ldots, \varphi_n)$ in V^*. This is nothing but a variational formulation for (4.4) which, therefore, has a unique solution. Thus, the entire sequence u_ε converges to its solution. □

The proof of Corollary 4.5 is an easy exercise which can safely be left to the reader (see Corollary 2.12 if necessary).

Proof of Theorem 4.6

This subsection is devoted to the proof of Theorem 4.6 concerning the $(n+1)$-scale convergence of gradients in perforated domains. The proof is similar in many respects to that of Theorem 2.6. However, since it is based on successive integrations by parts, a new difficulty arises here which is connected to the boundary terms produced by these integrations by parts. To get rid of them, we shall use test functions which vanish, as all their derivatives, near the holes boundaries.

To begin with, we introduce some notation which makes life easier when stating our results.

DEFINITION 4.8. A sequence φ_ε of $L^2(\Omega_\varepsilon)$ is said to be *bounded in* H_ε^{-1} if and only if there exists a positive constant c independent of ε such that

$$\left| \int_{\Omega_\varepsilon} \varphi_\varepsilon u\, dx \right| \leq c \| u \|_{H^1(\Omega_\varepsilon)} \quad \text{for any } u \in H^1(\Omega_\varepsilon).$$

In some sense, Definition 4.8 means that the sequence φ_ε is uniformly bounded in the dual space of $H^1(\Omega_\varepsilon)$.

We are now in a position to state two results on oscillating functions which are crucial for the proof of Theorem 4.6 (they are similar to Theorem 3.3 and Corollary 3.4).

THEOREM 4.9. *Let D^* be the set of smooth functions $\varphi(x, y_1, \ldots, y_n)$, periodic in $(y_1, \ldots, y_n)$, with compact support in Ω and compact support in the image of Y_k^* in the torus with respect to the variable y_k, for all $k \in \{1, \ldots, n\}$, i.e.*

$$D^* = \mathscr{D}[\Omega; \mathscr{D}_\#(Y_1^* \times \ldots \times Y_n^*)]. \tag{4.8}$$

For any $k \in \{1, \ldots, n\}$, let E_k^* be the subset of D^* composed of functions $\varphi(x, y_1, \ldots, y_n)$ with mean value zero with respect to the variable y_k, i.e.

$$E_k^* = \left\{ \varphi \in D^* \text{ such that} \int_{Y_k^*} \varphi\, dy_k = 0 \right\}. \tag{4.9}$$

Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are well separated (Definition 3.1). Then, for any *function φ in E_k^*,*

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}$$

(*see Definition* 4.8).

COROLLARY 4.10. *Let $\varphi \in \mathscr{D}[\Omega; \mathscr{D}_\#(Y_1^* \times \ldots \times Y_n^*)]$ be a function such that, for $k \in \{1, \ldots, n\}$, one has*

$$\int_{Y_k^*} \cdots \int_{Y_n^*} \varphi \, dy_k \ldots dy_n = 0. \tag{4.10}$$

Assume that the scales $\varepsilon_1, \ldots, \varepsilon_n$ are well separated; then

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}.$$

REMARK 4.11. We emphasise the difference between the set E_k^* defined above by (4.9) and the previous set E_k defined by (3.1): the elements of E_k^* depend on all variables $(y_1, \ldots, y_n)$ and not only on $(y_1, \ldots, y_k)$ because they must vanish on the holes boundaries at all scales.

Proof of Corollary 4.10. Fix $k \in \{1, \ldots, n\}$. Since the material parts Y_k^* are nonempty, there exist $(n - k + 1)$ functions $(\theta_k, \ldots, \theta_n)$ such that

$$\theta_j \in \mathscr{D}_\#(Y_j^*) \quad \text{and} \quad \int_{Y_j^*} \theta_j \, dy_j = 1 \quad \text{for } j \in \{k, \ldots, n\}.$$

Then any function φ satisfying (4.10) can be written as a sum of functions φ_j:

$$\varphi(x, y_1, \ldots, y_n) = \sum_{j=k}^{n} \varphi_j(x, y_1, \ldots, y_n),$$

where the functions φ_j are defined by the inductive formulae

$$\begin{cases} \varphi_n = \varphi - \theta_n \displaystyle\int_{Y_n} \varphi \, dy_n, \\ \varphi_j = (\theta_{j+1} \ldots \theta_n) \displaystyle\int_{Y_{j+1}} \cdots \int_{Y_n} \varphi \, dy_{j+1} \ldots dy_n \\ \qquad - (\theta_j \ldots \theta_n) \displaystyle\int_{Y_j} \cdots \int_{Y_n} \varphi \, dy_j \ldots dy_n \quad k \leqq j \leqq n - 1. \end{cases}$$

Each function φ_j belongs to the space E_j^* defined by (4.9) and Theorem 4.9 implies that

$$\frac{1}{\varepsilon_j}[\varphi_j]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}.$$

Hence,

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon = \sum_{j=k}^{n} \frac{\varepsilon_j}{\varepsilon_k} \frac{1}{\varepsilon_j}[\varphi_j]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}. \quad \square$$

As for Theorem 3.3, the key ingredient for proving Theorem 4.9 is the existence of 'special' solutions of the divergence equation:

$$\operatorname{div}_{y_k} \Psi = \varphi \quad \text{in } \Omega \times \prod_{k=1}^{n} Y_k^*. \tag{4.11}$$

More precisely, given a smooth right-hand side φ, having mean-value zero on Y_k^*, we seek a smooth solution Ψ, whose average is also zero on Y_k^*, and such that it vanishes on the holes boundaries at all scales. The following lemma provides such an existence result in the class E_k^* defined by (4.9). Its proof relies on an explicit integral representation of solutions of (4.11) (due to Bogowski [**11**]) and is a little tedious and technical (see the third subsection).

LEMMA 4.12. *For any* $k \in \{1, \ldots, n\}$ *and any function* $\varphi \in E_k^*$, *there exists a vector function* $\Psi \in (E_k^*)^n$ *such that*

$$\operatorname{div}_{y_k} \Psi = \varphi.$$

Furthermore, one can choose Ψ *in such a way that the application* S^* *defined by*

$$E_k^* \to (E_k^*)^N,$$
$$\varphi \mapsto S^*\varphi = \Psi,$$

is continuous with respect to the uniform continuity norm on $\Omega \times Y_1^* \times \ldots \times Y_n^*$.

Proof of Theorem 4.9. The proof proceeds by induction on k.

First step: $k = n$. Let $\varphi \in E_n^*$ (defined by (3.9)). Let T_ε^* be the application from E_n^* into E_n^* defined by

$$T_\varepsilon^* \varphi = \varepsilon_{n-1} \operatorname{div}_x S^*\varphi + \sum_{j=1}^{n-1} \frac{\varepsilon_{n-1}}{\varepsilon_j} \operatorname{div}_{y_j} S^*\varphi.$$

As in Theorem 3.3, by successive computations of the divergence of $S^*(T_\varepsilon^*)^p\varphi$, we obtain an equation similar to equation (3.5):

$$\frac{1}{\varepsilon_n}[\varphi]_\varepsilon = \sum_{p=0}^{m-1} (-1)^p \left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)^p \operatorname{div}[S^*(T_\varepsilon^*)^p\varphi]_\varepsilon + (-1)^m \left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)^m \frac{1}{\varepsilon_n}[(T_\varepsilon^*)^m\varphi]_\varepsilon. \tag{4.12}$$

However, from definition (4.9) of E_n^* and Definition 4.2 of Ω_ε, one has

$$S^*(T_\varepsilon^*)^p\varphi \in (E_n^*)^N \quad \text{and thus } [S^*(T_\varepsilon^*)^p\varphi]_\varepsilon \in \mathscr{D}(\Omega_\varepsilon).$$

We can now multiply equation (4.12) by any test function in $H^1(\Omega_\varepsilon)$ and integrate by parts with no contribution from the holes boundaries. Then, as in the proof of Theorem 3.3, it yields that each term div $[S^*(T_\varepsilon^*)^p\varphi]_\varepsilon$, and thus

$$\frac{1}{\varepsilon_n}[\varphi]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}$$

in the sense of Definition 4.8.

Second step: $k<n$. Assume that the result holds for all $j\in\{k+1,\dots,n\}$. Let $\varphi\in E_k^*$. From definition (4.11) of $S^*\varphi$, one has

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon=\operatorname{div}[S^*\varphi]_\varepsilon-\sum_{j=1}^{k-1}\frac{1}{\varepsilon_j}[\operatorname{div}_{y_j}S^*\varphi]_\varepsilon-\sum_{j=k+1}^{n}\frac{1}{\varepsilon_j}[\operatorname{div}_{y_j}S^*\varphi]_\varepsilon.$$

The function $\operatorname{div}_{y_j}S^*\varphi$ belongs to E_k^* for any $j\in\{1,\dots,k-1\}$ and to E_j^* for any $j\in\{k+1,\dots,n\}$. Then, from the induction assumption, the previous equality can be written

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon=\frac{\varepsilon_k}{\varepsilon_{k-1}}\frac{1}{\varepsilon_k}[\varphi_1^\varepsilon]_\varepsilon+\psi_1^\varepsilon,$$

where the function

$$\varphi_1^\varepsilon=-\sum_{j=1}^{k-1}\frac{\varepsilon_{k-1}}{\varepsilon_j}\operatorname{div}_{y_j}S^*\varphi$$

belongs to E_k^* and is bounded with respect to the uniform continuity norm, and the function

$$\psi_1^\varepsilon=\operatorname{div}[S^*\varphi]_\varepsilon-\sum_{j=k+1}^{n}\frac{1}{\varepsilon_j}[\operatorname{div}_{y_j}S^*\varphi]_\varepsilon\quad\text{is bounded in } H_\varepsilon^{-1}.$$

Reiterating m times yields

$$\frac{1}{\varepsilon_k}[\varphi]_\varepsilon=\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^m\frac{1}{\varepsilon_k}[\varphi_m^\varepsilon]_\varepsilon+\psi_m^\varepsilon,$$

where the function φ_m^ε belongs to E_k^* and is bounded with respect to the uniform continuity norm, and the function ψ_m^ε is bounded in H_ε^{-1}. This concludes the proof since the scales are well separated. □

We need two other lemmas concerning 'generalised' divergence-free functions of several variables and their orthogonal.

LEMMA 4.13. *For any $k\in\{1,\dots,n\}$, let D_k^* be the subset of $(D^*)^N$ (defined by (4.8)) composed of functions satisfying a 'generalised' divergence-free condition, i.e.*

$$\begin{cases} D_n^*=\{\Phi\in(D^*)^n;\ \operatorname{div}_{y_n}\Phi=0\},\\ D_k^*=\left\{\Phi\in(D^*)^N;\ \operatorname{div}_{y_n}\Phi=0 \text{ and } \int_{Y_{j+1}}\dots\int_{Y_n}\operatorname{div}_{y_j}\Phi=0\ \forall\, k\leqq j\leqq n-1\right\}.\end{cases}\tag{4.13}$$

These spaces have the following properties:

(i) *Any function $\bar\Phi(x,y_1,\dots,y_k)$ in $\mathscr{D}[\Omega;\mathscr{D}_\#(Y_1^*\times\dots\times Y_k^*)]^N$ can be expressed as the average of a function in D_k^*, i.e. there exists $\Phi(x,y_1,\dots,y_n)\in D_k^*$ such that*

$$\bar\Phi=\int_{Y_{k+1}}\dots\int_{Y_n}\Phi\,dy_{k+1}\dots dy_n.$$

(ii) *Any function $\bar\Phi(x)\in\mathscr{D}(\Omega)^N$ can also be expressed as the average of a function*

Φ in D_n^*, and, moreover, we have

$$\bar{\Phi} = \int_{Y_1} \ldots \int_{Y_n} \Phi \, dy_1 \ldots dy_n \quad \text{and} \quad \|\Phi\|_{L^2(\Omega \times Y_1^* \times \ldots \times Y_n^*)} \leqq c \|\bar{\Phi}\|_{L^2(\Omega)},$$

where the constant c is independent of Φ and $\bar{\Phi}$.

LEMMA 4.14. *Let H^* be the space of 'generalised' divergence-free functions in $L^2[\Omega; L^2_\#(Y_1^* \times \ldots \times Y_n^*)]^N$ defined by*

$$\Phi \in H^* \Leftrightarrow \begin{cases} \operatorname{div}_{y_n} \Phi = 0 & \text{in } Y_n^* \\ \Phi \cdot \nu = 0 & \text{on } \partial T_n \end{cases} \quad \text{and} \tag{4.14}$$

$$\begin{cases} \displaystyle\int_{Y_{k+1}} \ldots \int_{Y_n} \operatorname{div}_{y_k} \Phi = 0 & \text{in } Y_k^* \\ \displaystyle\int_{Y_{k+1}} \ldots \int_{Y_n} \Phi \cdot \nu = 0 & \text{on } \partial T_k \end{cases} \quad \forall\, 1 \leqq k \leqq n-1.$$

The subspace H^* has the following properties:

(i) *$(D^*)^N \cap H^*$ (see definition (4.5)) is dense into H^*.*

(ii) *The orthogonal of H^* is*

$$(H^*)^\perp = \left\{ \sum_{k=1}^{n} \nabla_{y_k} q_k(x, y_1, \ldots, y_k) \text{ with } q_k \in L^2[\Omega \times Y_1^* \times \ldots \times Y_{k-1}^*; H^1_\#(Y_k^*)/\mathbb{R}] \right\}. \tag{4.15}$$

REMARK 4.15. As seen in Remark 2.8, the 'generalised' divergence-free condition (4.14) includes a type of periodic boundary condition. Indeed, since a divergence-free function $\Psi \in L^2(Y_k^*)^N$ has a normal trace $\Psi \cdot \nu$ in $H^{-1/2}(\partial Y_k^*)$, the vector field Φ in (4.14) satisfies periodicity conditions on the part of ∂Y_k^* which intersects the boundary of the cube Y_k. These conditions are included in the 'generalised' divergence-free condition if equalities in (4.14) are taken in the sense of distributions in the set $E_\#(Y_k^*)$, or equivalently

$$\int_{Y_k^*} \ldots \int_{Y_n^*} \Phi \cdot \nabla_{y_k} \varphi \, dy_k \ldots dy_n = 0 \quad \text{for any } \varphi \in H^1_\#(Y_k) \quad \text{and} \quad 1 \leqq k \leqq n. \tag{4.16}$$

Then the periodicity conditions are

$$\begin{cases} \Phi \cdot \nu & \text{takes opposite values on opposite faces of } \partial Y_n \cap \partial Y_n^*, \\ \displaystyle\int_{Y_{k+1}} \ldots \int_{Y_n} \Phi \cdot \nu & \text{takes opposite values on opposite faces of } \partial Y_k \cap \partial Y_k^*, \quad 1 \leqq k \leqq n-1. \end{cases}$$

Proof of Theorem 4.6. Since the sequence u_ε is bounded in $H^1(\Omega_\varepsilon)$ (see estimate (4.2)), by application of Theorem 2.4, there exist two limits $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$ and

$\xi_0 \in L^2[\Omega \times Y_1 \times \ldots \times Y_n)^N$, such that, up to a subsequence,

$$\tilde{u}_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n) u_0,$$

$$\tilde{\nabla} u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(y_1, \ldots, y_n) \xi_0,$$

where $\tilde{}$ denotes the extension by zero outside Ω_ε and $\chi(y_1, \ldots, y_n)$ the multiscale characteristic function defined by Definition 4.2.

First step. Let us prove by induction that u_0 does not depend on the microscopic variables $(y_1, \ldots, y_n)$. Let $\Omega \in (D^*)^N$ (definition (4.8)). By integration by parts, one has

$$\varepsilon_n \int_{\Omega_\varepsilon} \nabla u_\varepsilon [\Phi]_\varepsilon \, dx = -\varepsilon_n \int_\Omega u_\varepsilon [\operatorname{div}_x \Phi]_\varepsilon \, dx$$
$$- \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_k} \Phi]_\varepsilon \, dx - \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_n} \Phi]_\varepsilon \, dx.$$

Then, passing to the $(n+1)$-scale limit in the previous equality yields

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} u_0 \operatorname{div}_{y_n} \Phi \, dx \, dy_1 \ldots dy_n = 0,$$

which proves that u_0 does not depend on y_n.

Assume that u_0 does not depend on the variables $(y_{k+1}, \ldots, y_n)$ for $k \in \{1, \ldots, n-1\}$. Let $\Phi \in D_{k+1}^*$ (definition (4.13)). By integration by parts, one has

$$\varepsilon_k \int_{\Omega_\varepsilon} \nabla u_\varepsilon [\Phi]_\varepsilon \, dx = -\varepsilon_k \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_x \Phi]_\varepsilon \, dx$$
$$- \sum_{j=1}^{k-1} \frac{\varepsilon_k}{\varepsilon_j} \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_j} \Phi]_\varepsilon \, dx - \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_k} \Phi]_\varepsilon \, dx$$
$$- \sum_{j=k+1}^{n-1} \frac{\varepsilon_k}{\varepsilon_j} \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_j} \Phi]_\varepsilon \, dx. \tag{4.17}$$

Since $\Phi \in D_j^*$ for any $j \in \{k+1, \ldots, n\}$, $\operatorname{div}_{y_j} \Phi$ satisfies assumption (4.10) in Corollary 4.10 with $j+1$. Thus,

$$\frac{1}{\varepsilon_{j+1}} [\operatorname{div}_{y_j} \Phi]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}$$

in the sense of Definition 4.8. Then, the last sum of (4.17) converges to 0 and passing to the $(n+1)$-scale limit in (4.17) yields

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_k^*} u_0 \operatorname{div}_{y_k} \bar{\Phi} \, dx \, dy_1 \ldots dy_k = 0,$$

$$\text{where } \bar{\Phi} = \int_{Y_{k+1}^*} \cdots \int_{Y_n^*} \Phi \, dy_{k+1} \ldots dy_n,$$

which implies, from part (i) of Lemma 4.13, that

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_k^*} u_0 \operatorname{div}_{y_k} \bar{\Phi}\, dx\, dy_1 \ldots dy_k = 0 \quad \forall \bar{\Phi} \in \mathscr{D}[\Omega; \mathscr{D}_\#(Y_1^* \times \ldots \times Y_k^*)]^N.$$

Hence function u_0 does not depend on variable y_k, which concludes the induction and proves that u_0 only depends on variable x.

Second step. Let us characterise $\chi(y_1, \ldots, y_n)\xi_0$ the $(n+1)$-scale limit of $\tilde{\nabla} u_\varepsilon$. Let $\Phi \in (D^*)^N \cap H^*$ be a smooth function satisfying the 'generalised' divergence-free condition (4.14). By integration by parts, one has

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon [\Phi]_\varepsilon\, dx = -\int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_x \Phi]_\varepsilon\, dx - \sum_{k=1}^{n-1} \frac{1}{\varepsilon_k} \int_{\Omega_\varepsilon} u_\varepsilon [\operatorname{div}_{y_k} \Phi]_\varepsilon\, dx \quad \text{since } \operatorname{div}_{y_n} \Phi = 0. \tag{4.18}$$

The function $\operatorname{div}_{y_k} \Phi$ satisfies the assumption (4.7) in Corollary 3.10 with $k+1$. Thus,

$$\frac{1}{\varepsilon_{k+1}} [\operatorname{div}_{y_k} \Phi]_\varepsilon \quad \text{is bounded in } H_\varepsilon^{-1}$$

in the sense of Definition 4.8. Then passing to the $(n+1)$-scale limit in (4.18) yields

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} \xi_0 \cdot \Phi\, dx\, dy_1 \ldots dy_n = -\int_\Omega u_0 \operatorname{div}_x \Theta\, dx$$
$$\text{where } \Theta = \int_{Y_1^*} \cdots \int_{Y_n^*} \Phi\, dy_1 \ldots dy_n, \tag{4.19}$$

since u_0 only depends on x. From part (ii) of Lemma 4.13, (4.19) implies that

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} \xi_0 \cdot \Phi\, dx\, dy_1 \ldots dy_n = -\int_\Omega u_0 \operatorname{div}_x \Theta\, dx$$
$$\forall \Theta \in \mathscr{D}(\Omega)^N \quad \text{with } \|\Phi\|_{L^2(\Omega \times Y_1^* \times \ldots \times Y_n^*)} \leqq c \|\Theta\|_{L^2(\Omega)},$$

which proves that $u_0 \in H^1(\Omega)$.

Another integration by parts on the right-hand side of (4.19), with respect to x, yields

$$\int_\Omega \int_{Y_1^*} \cdots \int_{Y_n^*} (\xi_0 - \nabla u_0) \cdot \Phi\, dy_1 \ldots dy_n = 0 \quad \text{for any } \Phi \in (D^*)^N \cap H^*.$$

By the density result of Lemma 4.14(i), the previous equality holds for any $\Phi \in H^*$, thus $\xi_0 - \nabla u_0 \in (H^*)^\perp$. In view of definition (4.15) of $(H^*)^\perp$, this proves that

$$\tilde{\nabla} u_\varepsilon \xrightarrow{(n+1)\text{-scale}} \chi(x, y_1, \ldots, y_n) \left[\nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, \ldots, y_k) \right]. \quad \square$$

Proof of technical lemmas

This subsection is devoted to the proofs of Lemmas 4.12, 4.13 and 4.14. We begin with Lemma 4.12, which is concerned with the existence of 'special' solutions of the equation $\operatorname{div}_{y_k} \Psi = \varphi$. Let us first remark that we can consider the variables

$(x, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$ as parameters. Therefore, we can simplify the notation by dropping the index k, and Lemma 4.12 reduces to:

LEMMA 4.16. *Let Y^* be an open subset of the unit cube Y, which satisfies Assumption 4.1. Let E^* be the set of smooth functions periodic in y with compact support in the image of Y^* in the torus and with mean value zero, i.e.*

$$E^* = \left\{\varphi \in \mathscr{D}_\#(Y^*) \text{ such that } \int_{Y^*} \varphi \, dy = 0\right\}. \tag{4.20}$$

Then, for any function $\varphi \in E^*$, there exists a vector-function $\Psi \in (E^*)^N$ such that

$$\operatorname{div} \Psi = \varphi. \tag{4.21}$$

Furthermore, we can choose Ψ such that it depends continuously, with respect to the uniform continuity norm, on φ.

The key ingredient of the proof of Lemma 4.16 is based on an integral representation, due to Bogowskii [11] (see also [4] for an equivalent result without integral representation), of smooth solutions of the equation $\operatorname{div} v = g$ in a bounded star-shaped domain ω:

PROPOSITION 4.17 (see [11]). *Let ω be a bounded domain of $\mathbf{R}^N$ which is star-shaped with respect to the origin and let K be a ball contained in ω and centred at the origin. Let $g \in \mathscr{D}(\omega)$ such that $\int_\omega g(x)\, dx = 0$. Then, for any function $q \in \mathscr{D}(K)$ such that $\int_K q(x)\, dx = 1$, the function v defined by*

$$v(x) = \int_\Omega g(y)(x-y)\left[\int_1^{+\infty} q(y + t(x-y))t^{N-1}\, dt\right] dy \tag{4.22}$$

belongs to $\mathscr{D}(\omega)^N$ and satisfies the equation $\operatorname{div} v = g$.

To apply Proposition 4.17 in the proof of Lemma 4.16, we shall divide Y^* in a union of star-shaped domains and φ in a corresponding sum of compactly supported functions with zero average. To this end, we also require the two following results:

LEMMA 4.18. *Let Y^* be an open subset of the unit cube Y, which satisfies Assumption 4.1. Then, there exists a matrix-function $W(y)$ such that*

$$W \in \mathscr{D}_\#(Y^*)^{N^2}, \quad \int_{Y^*} W(y)\, dy = Id \quad \text{and} \quad \operatorname{div}(W\xi) = 0 \quad \forall\, \xi \in \mathbf{R}^N. \tag{4.23}$$

Proof. Denoting by $(e_i)_{1 \leq i \leq N}$ the canonical basis of $\mathbf{R}^N$, we consider the following Stokes problem in the periodic unit cell:

$$\begin{cases} -\Delta_y w_i + \nabla_y p_i = e_i & \text{in } Y^*, \\ \operatorname{div}_y w_i = 0 & \text{in } Y^*, \\ w_i = 0 & \text{on } T, \\ y \mapsto p_i(y),\, w_i(y) & Y\text{-periodic}, \end{cases}$$

which has a unique solution w_i in $H^1_\#(Y^*)^N$. A well-known result in periodic homogenisation tells us that the matrix A defined by

$$Ae_i = \int_{Y^*} w_i(y)\,dy$$

is symmetric and positive definite since $E_\#(Y^*)$ is connected (if not, w_i could be zero). If the solution w_i were smooth and compactly supported away from the hole T, we could simply define the matrix-function

$$W(y)Ae_i = w_i(y) \quad \text{for } 1 \leqq i \leqq N.$$

However, this is not the case, and we need to use a density argument to approximate the solutions $(w_i)_{1\leqq i \leqq N}$ by divergence-free smooth functions with compact support. Then it is not difficult to construct the required matrix $W(y)$. We leave the details to the reader. □

LEMMA 4.19. *Let $(\omega_1, \dots, \omega_p)$ be p-connected domains of the unit torus such that their union ω is connected and let $(a_1, \dots, a_p)$ be p reals such that their sum is zero. Then, there exist p functions $(\tau_1, \dots, \tau_p)$ such that*

$$\sum_{k=1}^{p} \tau_k = 0 \quad \text{with } \tau_k \in \mathscr{D}_\#(\omega_k) \quad \text{and} \quad \int_{\omega_k} \tau_k\,dx = a_k \quad \text{for any } 1 \leqq k \leqq p. \tag{4.24}$$

Furthermore, we can choose each function τ_k such that it depends continuously, with respect to the uniform continuity norm, on $(a_1, \dots, a_p)$.

Proof. We proceed by induction on p. The result is obviously true for $p = 1$, and assuming it holds for the value $p-1$, we check it for p. Since ω is connected, it is easily seen that there must exist $p-1$ connected domains, for example $(\omega_1, \dots, \omega_{p-1})$ such that their union ω' is also connected (it is enough to consider the longest sequence of $k \leqq p-1$ domains such that their union is connected). With no loss of generality, we can assume that $\omega_{p-1} \cap \omega_p \neq \varnothing$.

From the induction assumption, there exist $p-1$ functions $(\tau'_1, \dots, \tau'_{p-1})$ such that

$$\begin{cases} \displaystyle\int_{\omega_k} \tau'_k\,dx = a_k & \text{if } 1 \leqq k \leqq p-2, \\ \displaystyle\int_{\omega_k} \tau'_k\,dx = a_{p-1} + a_p & \text{if } k = p-1, \end{cases} \quad \text{and} \quad \tau'_k \in \mathscr{D}_\#(\omega_k) \quad \textit{for all } 1 \leqq k \leqq p-1,$$

and since $\omega_{p-1} \cap \omega_p \neq \varnothing$, there exists $\tau' \in \mathscr{D}_\#(\omega_{p-1} \cap \omega_p)$ such that

$$\int_{\omega_p} \tau'\,dx = a_p.$$

Then, the functions $(\tau_1, \dots, \tau_p)$ defined by

$$\begin{cases} \tau_k = \tau'_k & \text{if } 1 \leqq k \leqq p-2, \\ \tau_k = \tau'_k - \tau' & \text{if } k = p-1, \\ \tau_k = \tau' & \text{if } k = p, \end{cases}$$

satisfy conditions (4.24) and it is clear that they depend linearly and continuously (with respect to the uniform continuity norm) on vector $(a_1, \dots, a_p)$, which concludes the proof. □

We are now in a position to prove Lemma 4.16.

Proof of Lemma 4.16. Since the image of Y^* in the unit torus is connected and has Lipschitz boundary, there exists a collection of star-shaped subdomains $(\omega_1, \dots, \omega_p)$ in the unit torus (see for example [9]) such that

$$Y^* = \bigcup_{k=1}^{p} \omega_k.$$

Then, for any $\varphi \in \mathscr{D}_\#(Y^*)$ (which by definition is compactly supported away from the hole T), there exists a partition of the unity $(\theta_1, \dots, \theta_p)$ such that

$$\theta_k \in \mathscr{D}_\#(Y^*), \quad \theta_{k|\omega_k} \in \mathscr{D}_\#(\omega_k) \quad \text{and} \quad \sum_{k=1}^{p} \theta_k = 1 \text{ in the support of } \varphi.$$

Next, we define a collection of reals $(a_1, \dots, a_p)$ such that

$$a_k = \int_{\omega_k} \theta_k \varphi \, dy.$$

Clearly, we have

$$\sum_{k=1}^{p} a_k = 0 \quad \text{if and only if} \int_{Y^*} \varphi \, dy = 0.$$

In this case, by application of Lemma 4.19, there exists a collection of functions $(\tau_1, \dots, \tau_p)$ satisfying (4.24). Thus, the functions $(\varphi_1, \dots, \varphi_p)$ defined by

$$\varphi_k = \theta_k \varphi - \tau_k$$

satisfy

$$\sum_{k=1}^{p} \varphi_k = \varphi, \quad \varphi_k \in \mathscr{D}_\#(\omega_k) \quad \text{and} \quad \int_{\omega_k} \varphi_k \, dy = 0 \quad \text{for any } 1 \leqq k \leqq p.$$

To each function φ_k, we now apply Proposition 4.17 and, summing up the results, we obtain the existence of a function $\Phi \in \mathscr{D}_\#(Y^*)^N$ such that

$$\operatorname{div} \Phi = \varphi \quad \text{in } Y^*. \tag{4.25}$$

However, Φ is not necessarily of average zero in Y^*. We can remedy this inconvenience thanks to Lemma 4.18. Indeed, the function Ψ defined by

$$\Psi = \Phi - W\left(\int_{Y^*} \Phi \, dy \right)$$

satisfies the same equation (4.25) and has mean-value zero by virtue of (4.23). Finally, remarking that all the intermediate functions depend continuously on φ, so does Ψ, which concludes the proof of Lemma 4.16. □

We now give the proof of Lemma 4.13 concerning the attainability of any smooth functions as average of 'generalised' divergence-free functions in D^*.

Proof of Lemma 4.13. (i) Let $\bar{\Phi}(x, y_1, \dots, y_k) \in \mathscr{D}[\Omega; \mathscr{D}_\#(Y_1^* \times \dots \times Y_k^*)]^N$. For any

$j \in \{1, \ldots, n\}$, let $W_j(y_j)$ be the matrix-valued function defined by (4.23) with $Y = Y_j$. Then, the function defined by

$$\Phi(x, y_1, \ldots, y_n) = W_n(y_n) \ldots W_{k+1}(y_{k+1}) \bar{\Phi}(x, y_1, \ldots, y_k)$$

satisfies the first property.

(ii) The function defined by

$$\Phi(x, y_1, \ldots, y_n) = W_n(y_n) \ldots W_1(y_1) \bar{\Phi}(x)$$

satisfies the second property. □

We finish this subsection with the proof of Lemma 4.14 on the space H^* of 'generalised' divergence-free functions.

Proof of Lemma 4.14. (i) Let $\Phi \in H^*$, and, for any $k \in \{1, \ldots, n\}$, let $\Phi_k(x, y_1, \ldots, y_k) \in L^2(\Omega \times Y_1^* \times \ldots \times Y_k^*)^N$ a family of functions defined by the following induction:

$$\begin{cases} \Phi_n = \Phi, \\ \Phi_k = \displaystyle\int_{Y_{k+1}^*} \ldots \int_{Y_n^*} \Phi \, dx \, dy_1 \ldots dy_n = 0 \quad \textit{for } 1 \leqq k \leqq n-1. \end{cases} \tag{4.26}$$

Then, by a classical argument each of these functions can be approached by a smooth divergence-free sequence, i.e. for any $k \in \{1, \ldots, n\}$, there exists a sequence $\{\Phi_k^\delta\}_{\delta > 0}$ which satisfies

$$\begin{gathered} \Phi_k^\delta \in \mathscr{D}[\Omega; \mathscr{D}_\#(Y_1^* \times \ldots \times Y_k^*)]^N, \\ \operatorname{div}_{y_k} \Phi_k^\delta = 0 \quad \text{and} \quad \lim_{\delta \to 0} \| \Phi_k^\delta - \Phi_k \|_{L^2(\Omega \times Y_1^* \times \ldots \times Y_k^*)} = 0. \end{gathered} \tag{4.27}$$

For any $j \in \{1, \ldots, n\}$, let $W_j(y_j)$ be the matrix-valued function defined by (4.23) with $Y = Y_j$, and let $\Phi^\delta(x, y_1, \ldots, y_n)$ be the function defined by

$$\Phi^\delta = \Phi_n^\delta + \sum_{k=1}^{n-1} (W_n \ldots W_{k+1}) \left(\Phi_k^\delta - \int_{Y_{k+1}^*} \Phi_{k+1}^\delta \right).$$

It is clear that $\Phi^\delta \in (D^*)^N$. Furthermore, (4.26) and (4.27) imply that

$$\lim_{\delta \to 0} \| \Phi^\delta - \Phi \|_{L^2(\Omega \times Y_1^* \times \ldots \times Y_n^*)} = 0.$$

An easy computation shows that

$$\operatorname{div}_{y_k} \left(\int_{Y_{k+1}^*} \ldots \int_{Y_n^*} \Phi^\delta \, dy_{k+1} \ldots dy_n \right) = 0.$$

Thus, $\Phi^\delta \in (D^*)^N \cap H^*$ is an approximation of $\Phi \in H^*$ in $L^2[\Omega; L_\#^2(Y_1^* \times \ldots \times Y_n^*)]^N$, which concludes the proof of part (i).

(ii) Remark that

$$H^* = \bigcap_{k=1}^{n} H_k^*$$

where H_k^* is the subset of $L^2[\Omega; L_\#^2(Y_1^* \times \ldots \times Y_n^*)]^N$ composed of functions Φ such that

$$\begin{cases} \operatorname{div}_{y_n} \Phi = 0 & \text{in } Y_n^*, \\ \Phi \cdot \nu = 0 & \text{on } \partial T_n, \end{cases} \quad \text{if } k = n, \qquad \text{or}$$

$$\begin{cases} \int_{Y_{k+1}} \dots \int_{Y_n} \operatorname{div}_{y_k} \Phi = 0 & \text{in } Y_k^*, \\ \int_{Y_{k+1}} \dots \int_{Y_n} \Phi \cdot \nu = 0 & \text{on } \partial T_k, \end{cases} \quad \text{if } 1 \leqq k \leqq n-1.$$

Since the boundary of each $E_\#(Y_k^*)$ is Lipschitz (Assumption 4.1), the orthogonal of H_k^* is

$$(H_k^*)^\perp = \{\nabla_{y_k} q(x, y_1, \dots, y_k) \quad \text{with} \quad q \in L^2[\Omega \times Y_1^* \times \dots \times Y_{k-1}^*; H_\#^1(Y_k^*)]\}.$$

As in the proof of Lemma 3.7, it is thus enough to prove that

$$\sum_{k=1}^{n} H_k^*$$

is closed to obtain the desired result (4.15). Let $\Phi \in L^2(\Omega \times Y_1^* \times \dots \times Y_n^*)^N$; one can write, owing to (4.23),

$$\Phi = W_n(y_n) \int_{Y_n^*} \Phi \, dy_n + \left(\Phi - W_n(y_n) \int_{Y_n^*} \Phi \, dy_n \right) \in H_n^* + H_{n-1}^*.$$

Hence,

$$H_n^* + H_{n-1}^* = L^2(\Omega \times Y_1^* \times \dots \times Y_n^*)^N = \sum_{k=1}^{n} H_k^* \quad \text{is closed,}$$

which ends the proof of part (ii). □

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