Homogenization and Localization with an Interface

GRÉGOIRE ALLAIRE, YVES CAPDEBOSCQ &
ANDREY PIATNITSKI

ABSTRACT. We consider the homogenization of a spectral problem for a diffusion equation posed in a singularly perturbed periodic medium. Denoting by ε the period, the diffusion coefficients are scaled as ε^2. The domain is composed of two periodic medium separated by a planar interface, aligned with the periods. Three different situations arise when ε goes to zero. First, there is a global homogenized problem as if there were no interface. Second, the limit is made of two homogenized problems with a Dirichlet boundary condition on the interface. Third, there is an exponential localization near the interface of the first eigenfunction.

1. INTRODUCTION

This paper is devoted to the homogenization of the eigenvalue problem for a singularly perturbed diffusion equation in a periodic medium with an interface. Denoting by ε the period, the diffusion coefficient is assumed to be of the order of ε^2. For simplicity we suppose the domains to be cylindrical of the form $\Omega = \mathbb{T}^{n-1} \times [-\ell, L]$ where $\mathbb{T}^{n-1}$ is the unit torus $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. We consider the following model

$$
(1.1)\qquad \begin{cases} -\varepsilon^2 \operatorname{div}\left(a\left(\dfrac{x}{\varepsilon}, x_n\right) \nabla \phi^\varepsilon\right) + \Sigma\left(\dfrac{x}{\varepsilon}, x_n\right) \phi^\varepsilon \\ \qquad\qquad = \lambda^\varepsilon \sigma\left(\dfrac{x}{\varepsilon}, x_n\right) \phi^\varepsilon \quad \text{in } \Omega, \\ \phi^\varepsilon(\cdot, -\ell) = 0 \quad \text{and} \quad \phi^\varepsilon(\cdot, L) = 0, \\ (x_1, \dots, x_{n-1}) \to \phi^\varepsilon(x), \quad \mathbb{T}^{n-1}\text{-periodic}, \end{cases}
$$

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where λ^ε, ϕ^ε are an eigenvalue and an eigenfunction (throughout this paper, the eigenfunctions are normalized by $\|\phi^\varepsilon\|_{L^2(\Omega)} = 1$). In (1.1) the coefficients are periodic of period $[0,1]^n$ with respect to the fast variable $y = x/\varepsilon$. We introduce the two sub-domains $\Omega_1 = \mathbb{T}^{n-1} \times (-\ell, 0)$ and $\Omega_2 = \mathbb{T}^{n-1} \times (0, L)$ separated by an interface located at $x_n = 0$, the hyperplane $\Gamma = \mathbb{T}^{n-1} \times \{0\}$. Denoting by $\chi_i(x_n)$ the characteristic function of Ω_i (satisfying $\chi_1 + \chi_2 = 1$ and $\chi_1\chi_2 = 0$ in Ω), the coefficients are assumed to be given as

$$\begin{cases} a(y, x_n) = \chi_1(x_n)a_1(y) + \chi_2(x_n)a_2(y), \\ \Sigma(y, x_n) = \chi_1(x_n)\Sigma_1(y) + \chi_2(x_n)\Sigma_2(y), \\ \sigma(y, x_n) = \chi_1(x_n)\sigma_1(y) + \chi_2(x_n)\sigma_2(y). \end{cases} \tag{1.2}$$

The periodic boundary conditions with respect to the variables $(x_1, \ldots, x_{n-1})$ (tangential to the interface) is not crucial but simplifies the exposition (all our results would hold for Dirichlet or Neumann boundary conditions). Problem (1.1) is supposed to be uniformly elliptic and self-adjoint so that it admits a countable infinite number of non-trivial solutions $(\lambda_m^\varepsilon, \phi_m^\varepsilon)_{m\ge1}$. The precise assumptions on the coefficients of (1.1) are given in Section 2. In any case, by standard regularity results, each eigenfunction ϕ_m^ε is continuous, and by virtue of the Krein-Rutman theorem the first eigenvalue λ_1^ε is simple and the corresponding eigenfunction ϕ_1^ε can be chosen positive. Because of this property, the first eigenpair has a special physical signification, and we are mostly interested in its behavior as ε goes to zero, although the case of higher level eigenpairs is also treated in some occasions.

The motivation for studying this model comes from several applications. First, it can be seen as a semi-classical limit problem for a Schrödinger-type equation with periodic potential, as well as periodic metric. As is well known, the long time behavior (for times of order ε^{-2}) of the corresponding parabolic equation is governed by the first eigenpair of (1.1): this is the so-called ground-state asymptotic problem (see, e.g. [21], [29]). Second, it plays an important role in the uniform controllability of the corresponding wave equation (see, e.g. [16]). Third, (1.1) is a model of a reaction-diffusion equation which is used for determining the power distribution in a nuclear reactor core. This is the so-called criticality problem for the one-group neutron diffusion equation (for more details, we refer to [2] and references therein). In this last application the homogenization results for (1.1) are at the basis of many multiscale-type numerical methods for computing its solutions (see, e.g. [14] and references therein). There are other works in the literature concerned with the effect of interfaces in homogenization theory (see e.g. [8], [9–11], Chapter 9 in [12]). However, these previous works focus on a different scaling of (1.1), namely without the ε^2 factor in front of the diffusion operator.

The homogenization of (1.1) is classical in the case of purely periodic coefficients, i.e., depending only on the fast variable x/ε [5], [2]. When the coefficients depend smoothly on the slow variable x (which is not the case here), the asymptotic of (1.1) is also partly understood [6] (see also [4], [29] for related results). However, in most applications, the coefficients actually depend on the slow variable x in a non-smooth manner, since they usually exhibit jumps at material interfaces. This makes model (1.1) with assumptions (1.2) physically relevant. Such an "interface" model has already been studied by two of the authors [3] in one space dimension.

The limit behavior of (1.1) is mainly governed by the first eigenpair (ψ_1, μ_1) in the unit cell of Ω_1, and (ψ_2, μ_2) in the unit cell of Ω_2, solutions of

$$(1.3)\qquad \begin{cases} -\operatorname{div}(a_i(y)\nabla\psi_i) + \Sigma_i(y)\psi_i = \mu_i\sigma_i(y)\psi_i & \text{in } \mathbb{T}^n, \\ y \to \psi_i(y)\,, & \mathbb{T}^n\text{-periodic and positive} \end{cases}$$

$i = 1, 2$. Before we explain our results, let us recall the result of [5] in the purely periodic case, namely when $a_1 = a_2$, $\Sigma_1 = \Sigma_2$, and $\sigma_1 = \sigma_2$. Asymptotically, each eigenfunction ϕ^ε is the product of the oscillatory term $\psi_1(x/\varepsilon)$ and of an eigenfunction for an homogenized spectral problem (we call this a factorization principle).

Theorem 1.1 ([5]). *Assuming that $a_2 = a_1$, $\Sigma_2 = \Sigma_1$, and $\sigma_2 = \sigma_1$, the m^{th} eigenpair $(\lambda_m^\varepsilon, \phi_m^\varepsilon)$ of (1.1) satisfies*

$$\phi_m^\varepsilon(x) = \psi_1\left(\frac{x}{\varepsilon}\right) u_m^\varepsilon(x) \quad \text{and} \quad \lambda_m^\varepsilon = \mu_1 + \varepsilon^2\nu_m + o(\varepsilon^2),$$

where, up to a subsequence, the sequence u_m^ε converges weakly in $H^1(\Omega)$ to u_m, and (ν_m, u_m) is the m^{th} eigenvalue and eigenvector for the homogenized problem

$$\begin{cases} -\operatorname{div}(\bar{D}\nabla u_m) = \nu_m\bar{\sigma}u_m & \text{in } \Omega, \\ u_m(\cdot, -\ell) = u_m(\cdot, L) = 0, \\ (x_1, \dots, x_{n-1}) \to u_m(x)\,, & \mathbb{T}^{n-1}\text{-periodic.} \end{cases}$$

The homogenized coefficients are given by

$$(1.4)\qquad \begin{aligned} \bar{D}_{ij} &= \int_{\mathbb{T}^n} \psi_1^2(y)a_1(y)(\nabla\xi_i + e_i)\cdot(\nabla\xi_j + e_j)\,dy\,, \\ \bar{\sigma} &= \int_{\mathbb{T}^n} \sigma_1(y)\psi_1^2(y)\,dy\,, \end{aligned}$$

where the function ξ_i, for $1 \le i \le n$, is the solution of

$$(1.5)\qquad \begin{cases} -\operatorname{div}(\psi_1^2(y)a_1(y)(\nabla\xi_i + e_i)) = 0 & \text{in } \mathbb{T}^n, \\ y \to \xi_i(y)\,, & \mathbb{T}^n\text{-periodic.} \end{cases}$$

In the case of smooth coefficients with respect to the slow variable x, we now recall the results of [6]. The unit cell problem (1.3) is now parametrized by the point $x \in \Omega$ and we denote by $(\mu(x), \psi(y,x))$ its first positive eigenpair.

Theorem 1.2 ([6]). *Assume that $a(y,x)$, $\Sigma(y,x)$, and $\sigma(y,x)$ are of class C^2, and that the cell eigenvalue $\mu(x)$ admits a unique non-degenerate minimum $x_0 \in \Omega$. Then, the m^{th} eigenpair $(\lambda_m^\varepsilon, \phi_m^\varepsilon)$ of (1.1) satisfies*

$$\phi_m^\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}, x\right) u_m^\varepsilon\left(\frac{x - x_0}{\sqrt{\varepsilon}}\right) \quad \text{and} \quad \lambda_m^\varepsilon = \mu(x_0) + \varepsilon \nu_m + o(\varepsilon),$$

where, up to a subsequence and a multiplicative factor (for renormalization), the sequence $u_m^\varepsilon(z)$ converges weakly in $H^1(\mathbb{R}^n)$ to $u_m(z)$, and (ν_m, u_m) is the m^{th} eigenvalue and eigenvector for the homogenized problem

$$-\operatorname{div}_z(\bar{D}\nabla_z u_m) + \left(\frac{1}{2}\nabla\nabla\mu(x_0)z \cdot z + \bar{c}\right) u_m = \nu_m \bar{\sigma} u_m \quad \text{in } \mathbb{R}^n.$$

The homogenized coefficients are given by formula (1.4) and (1.5) evaluated at x_0.

It is worth pointing out the main differences between Theorems 1.1 and 1.2. First, the corrector term in the ansatz for the eigenvalue is not of the same order in both cases. Second, there is a localization phenomenon at the scale $\sqrt{\varepsilon}$ in Theorem 1.2.

The situation considered in the present paper is intermediate between those of Theorems 1.1 and 1.2. It turns out that several different limit behaviors of the eigensolutions, as ε tends to zero, can occur depending on a criterion, defined by (2.18) and (3.1). We omit momentarily to define precisely this selection criterion since it requires the introduction of additional variational problems as it will be shown in the next section. There are four different limit behaviors of the eigensolutions, some of them being very close to that of Theorem 1.1 (i.e., the interface is "transparent"), and some others featuring a localization phenomenon at the interface in the spirit of Theorem 1.2.

The first limiting case, which can occur only when $\mu_1 = \mu_2$, can be interpreted as a "transparent" interface. Indeed, the second part of Theorem 3.1 shows that there is still a factorization principle, namely there exists a function $\psi(y)$ converging away from the interface to the cell eigensolutions ψ_1 and ψ_2 of (1.3) such that

$$(1.6) \qquad \phi_m^\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right) u_m^\varepsilon(x) \quad \text{and} \quad \lambda_m^\varepsilon = \mu_2 + \varepsilon^2 \nu_m + o(\varepsilon^2),$$

where u_m^ε converges weakly to u_m in $H^1(\Omega)$, and $(\lambda_m, u_m)_{m\geq 1}$ are the eigenpairs of the homogenized problem

$$\begin{cases} -\operatorname{div}((\chi_1(x)\bar{D}_1 + \chi_2(x)\bar{D}_2)\nabla u) = \nu(\chi_1(x)\bar{\sigma}_1 + \chi_2(x)\bar{\sigma}_2)u & \text{in } \Omega, \\ u(\cdot, -\ell) = 0 \quad \text{and} \quad u(\cdot, L) = 0, & \\ (x_1, \dots, x_{n-1}) \to u(x), & \mathbb{T}^{n-1}\text{-periodic}, \end{cases}$$

where the homogenized coefficients are computed by formulas similar to (1.4) and (1.5). This case is a simple extension of the purely periodic case since the interface does not affect the type of limit problem.

The second limiting case, which can occur only when $\mu_1 \neq \mu_2$ (without loss of generality, we always assume $\mu_1 \geq \mu_2$), is when the eigenfunctions concentrate on one half domain and vanish in the other half. The same factorization (1.6) (with a slightly different function ψ) takes place, but the homogenized problem is limited to Ω_2 (see Theorem 3.4)

$$u \equiv 0 \text{ in } \Omega_1 \quad \text{and} \quad \begin{cases} -\operatorname{div}(\bar{D}_2\nabla u) = \nu\bar{\sigma}_2 u & \text{in } \Omega_2, \\ u(\cdot, 0) = 0 \quad \text{and} \quad u(\cdot, L) = 0, & \\ (x_1, \dots, x_{n-1}) \to u(x), & \mathbb{T}^{n-1}\text{-periodic}. \end{cases}$$

The third limiting case, which can occur only when $\mu_1 = \mu_2$, corresponds to a "repulsive" interface. Asymptotically each half domain tends to separate, and the homogenized problem is posed on the two disconnected subdomains, with a Dirichlet condition on the interface. The same factorization (1.6) takes place but the homogenized problem is (see Theorem 3.4)

$$\begin{cases} -\operatorname{div}(\bar{D}_1\nabla u) = \nu\bar{\sigma}_1 u & \text{in } \Omega_2, \\ u(\cdot, -\ell) = 0 \quad \text{and} \quad u(\cdot, 0) = 0, & \\ (x_1, \dots, x_{n-1}) \to u(x), & \mathbb{T}^{n-1}\text{-periodic}, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(\bar{D}_2\nabla u) = \nu\bar{\sigma}_2 u & \text{in } \Omega_2, \\ u(\cdot, 0) = 0 \quad \text{and} \quad u(\cdot, L) = 0, & \\ (x_1, \dots, x_{n-1}) \to u(x), & \mathbb{T}^{n-1}\text{-periodic}. \end{cases}$$

Finally, the interface can induce a drastic change in the type of limit problem, since the first eigenfunction concentrates exponentially fast at the interface. In the fourth limiting case, there is no factorization principle as above, but rather a localization principle at the discontinuity (see Theorem 3.5). The first eigenvalue λ_1^ε

converges to a limit λ_1 which is below the cell eigenvalues, $0 < \lambda_1 < \min(\mu_1, \mu_2)$, and the convergence is exponential in the sense that there exists $\tau > 0$ such that

$$|\lambda_1^\varepsilon - \lambda_1| \le C \exp\left(-\frac{\tau}{\varepsilon}\right),$$

whereas the first normalized eigenvector satisfies

$$\left\|\nabla\phi_1^\varepsilon(x) - \frac{1}{\sqrt{\varepsilon}}\nabla\left(\Psi\left(\frac{x}{\varepsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\varepsilon(x) - \frac{1}{\sqrt{\varepsilon}}\Psi\left(\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \le C \exp\left(-\frac{\tau}{\varepsilon}\right).$$

The limit function $\Psi(y)$ decreases exponentially away from the interface, and (λ_1, Ψ) is the first eigenpair of an equation posed in an infinite strip

$$-\operatorname{div}(a(y, y_n)\nabla\Psi) + \Sigma(y, y_n)\Psi = \lambda_1\sigma(y, y_n)\Psi \quad \text{in } \mathbb{T}^{n-1} \times (-\infty, \infty).$$

This paper is organized as follows. Our main results and the precise definition of the situations described above are given in Section 3. Previously, in Section 2 we introduce our notation and the auxiliary variational problems that are crucial to the statement of our main results. Section 4 contains the proofs corresponding to the situations when homogenization takes place without localization. Section 5 is devoted to the proof of our results when a localization phenomenon occurs. Eventually Section 6 is concerned with an auxiliary interface variational problem which is at the basis of the proposed selection criterion between the different cases.

2. NOTATION AND AUXILIARY VARIATIONAL PROBLEMS

We first introduce the notation and assumptions used throughout this paper. Our assumptions on the coefficients of problem (1.1) are as follows.

All functions $(a_{1,ij})_{1\le i,j\le n}$, $(a_{2,ij})_{1\le i,j\le n}$, Σ_1, Σ_2, σ_1, and σ_2 are assumed to be measurable, $[0,1]^n$ periodic, and bounded. The coefficients Σ_1, Σ_2, σ_1, σ_2 are also bounded from below by positive constants. The diffusion matrices a_1 and a_2 are symmetric $n \times n$ matrices, assumed to be coercive, i.e., there exists a constant $C > 0$ such that for any $\xi \in \mathbb{R}^n$,

$$a_1(y)\xi\cdot\xi \ge C|\xi|^2 \quad \text{and} \quad a_2(y)\xi\cdot\xi \ge C|\xi|^2, \quad \text{for a.e. } y \in [0,1]^n.$$

Moreover, we assume that all coefficients of (1.1) have a minimal regularity in order that all the solutions involved possess $W^{1,\infty}$ regularity. For instance, if all the coefficients are of Hölder class C^γ, $\gamma > 0$, then this condition is satisfied, see [19]. In particular, this assumption is crucial to state that the discontinuity function α, defined by (2.18), is bounded in L^∞.

The domain Ω under consideration is $\Omega = \mathbb{T}^{n-1} \times [-\ell, L]$. The right-hand-side and left-hand-side sub-domains are $\Omega_1 = \mathbb{T}^{n-1} \times (-\ell, 0)$ and $\Omega_2 = \mathbb{T}^{n-1} \times (0, L)$. The interface is $\Gamma = \Omega \cap \{x_n = 0\}$. We also define the infinite strip

$G = \mathbb{T}^{n-1} \times (-\infty, +\infty)$, which has an interface $\Gamma = G \cap \{x_n = 0\}$ (we give the same name to the interface in G and Ω). The two left and right semi-infinite strips are noted $G_1 = G \cap \{x_n < 0\}$ and $G_2 = G \cap \{x_n > 0\}$. For all $x = (x_1, x_2, \dots, x_n) \in \Omega$, we note $x' = (x_1, \dots, x_{n-1})$ the first $n-1$ coordinates of x, living in $\mathbb{T}^{n-1}$. We therefore write $x = (x', x_n)$. We use the same notation for $y = (y', y_n) \in G$. The solutions of problem (1.1) belongs to the space $H^1_{\#,0}(\Omega) = \{u \in H^1(\Omega) \text{ s.t. } x' \to u(x', x_n) \text{ is } \mathbb{T}^{n-1} \text{ periodic and } u(x', -\ell) = u(x', L) = 0\}$. Also, to make the micro and macro periods consistent, we assume that $\varepsilon = 1/k$ with an integer k, $k \to \infty$.

Let us now introduce the cell problems which will govern the limit behavior of (1.1). At the difference of the one dimensional case (see [3]), it is not possible to choose a normalization condition of the cell eigenfunctions ψ_1 and ψ_2, solutions of (1.3), such that $\psi_1 = \psi_2$ on the interface Γ. To connect continuously ψ_1 in Ω_1 and ψ_2 in Ω_2, we need to introduce boundary layers $N_{1,0}$ and $N_{2,0}$ given by

$$\text{(2.1a)} \qquad \begin{cases} -\operatorname{div}\left(\psi_1^2 a_1(y) \nabla N_{1,0}\right) = 0 & \text{in } G_1, \\ N_{1,0}(\cdot, y_n), & \mathbb{T}^{n-1}\text{-periodic}, \\ N_{1,0}(\cdot, 0) = \dfrac{\psi_2}{\psi_1}(\cdot, 0), & \end{cases}$$

and

$$\text{(2.1b)} \qquad \begin{cases} -\operatorname{div}(\psi_2^2 a_2(y) \nabla N_{2,0}) = 0 & \text{in } G_2, \\ N_{2,0}(\cdot, y_n), & \mathbb{T}^{n-1}\text{-periodic}, \\ N_{2,0}(\cdot, 0) = \dfrac{\psi_1}{\psi_2}(\cdot, 0). & \end{cases}$$

Then, Corollary 2.5 states that the function ψ_0 defined by

$$\text{(2.2)} \qquad \psi_0(y) = \begin{cases} \psi_1(y)(1 + N_{1,0}(y)) & \text{for } y_n < 0, \\ \psi_2(y)(1 + N_{2,0}(y)) & \text{for } y_n > 0, \end{cases}$$

belongs to $H^1(G)$, is continuous through the interface Γ, and is an eigenfunction in G_1 and in G_2. In the generic case when $\mu_1 \neq \mu_2$ (without loss of generality we assume that $\mu_1 \geq \mu_2$) the function ψ_0 would correspond to two different eigenvalues on each side of the interface Γ. Clearly, only the smallest one has a chance to appear in the limiting process. For this reason we use an alternative cell eigensolution, described in the following lemma (for a proof, see [15]).

Lemma 2.1. *For any $\theta \in \mathbb{R}$, there exists a first normalized eigenpair $(\mu_1(\theta), \psi_{1,\theta})$ of the eigenvalue problem*

$$\text{(2.3)} \qquad \begin{cases} -\operatorname{div}(a_1(y) \nabla \psi_{1,\theta}) + \Sigma_1(y) \psi_{1,\theta} = \mu_1(\theta) \sigma_1(y) \psi_{1,\theta} & \text{in } [0,1]^n, \\ \psi_{1,\theta}(y) e^{-\theta y_n}, & \mathbb{T}^n\text{-periodic}, \end{cases}$$

where y_n is the n^{th} coordinate of y (normal to the interface). The first eigenvalue $\mu_1(\theta)$ is simple and the first eigenfunction $\psi_{1,\theta}$ can be chosen positive. Furthermore, the map $\theta \to \mu_1(\theta)$ is a strictly concave function reaching its maximum at $\theta = 0$. Thus, if $\mu_1 \equiv \mu_1(0) > \mu_2 \equiv \mu_2(0)$, there exists a unique $\theta > 0$ such that $\mu_1(\theta) = \mu_2$.

Remark that $\psi_{1,\theta}$, extended to G_1, is the product of a periodic function and of an exponentially decreasing function $\exp(\theta y_n)$. As before, it is not possible in general to connect continuously $\psi_{1,\theta}$ in G_1 and ψ_2 in G_2. To ensure continuity on Γ, we also introduce boundary layers defined by

$$\begin{cases} -\operatorname{div}(\psi_{1,\theta}^2 a_1(y)\nabla N_1) = 0 & \text{in } G_1, \\ N_1(\cdot, y_n), & \mathbb{T}^{n-1}\text{-periodic}, \\ N_1(\cdot, 0) = \dfrac{\psi_2}{\psi_{1,\theta}}(\cdot, 0), & \end{cases} \tag{2.4a}$$

and

$$\begin{cases} -\operatorname{div}(\psi_2^2 a_2(y)\nabla N_2) = 0 & \text{in } G_2, \\ N_2(\cdot, y_n), & \mathbb{T}^{n-1}\text{-periodic}, \\ N_2(\cdot, 0) = \dfrac{\psi_{1,\theta}}{\psi_2}(\cdot, 0). & \end{cases} \tag{2.4b}$$

Proposition 2.2. *The function ψ defined by*

$$\psi(y) = \begin{cases} \psi_{1,\theta}(y)(1 + N_1(y)) & \text{for } y_n \le 0, \\ \psi_2(y)(1 + N_2(y)) & \text{for } y_n > 0, \end{cases} \tag{2.5}$$

belongs to $H^1(G)$, is continuous through the interface Γ, and satisfies, for $i = 1, 2$ (with the same eigenvalue μ_2),

$$-\operatorname{div}(a_i \nabla\psi) + \Sigma_i \psi = \mu_2 \sigma_i \psi \quad \text{in } G_i. \tag{2.6}$$

Furthermore, there exist three positive constants $\tau > 0$, $c_1 > 1$ and $c_2 > 1$ such that

$$\begin{aligned} \lim_{y_n \to -\infty} e^{-\tau y_n} e^{-\theta y_n} |\nabla(\psi(y) - c_1 \psi_{1,\theta}(y))| &= 0, \\ \lim_{y_n \to \infty} e^{\tau y_n} |\nabla(\psi(y) - c_2 \psi_2(y))| &= 0. \end{aligned} \tag{2.7}$$

As a consequence, there exists a positive constant $C > 0$ such that

$$\frac{1}{C} \le e^{-\theta y_n}\psi(y) \le C \text{ for } y_n < 0 \quad \text{and} \quad \frac{1}{C} \le \psi(y) \le C \text{ for } y_n > 0. \tag{2.8}$$

The mapping

$$T : H^1_{\#,0}(\Omega) \rightarrow H^1_{\#,0}(\Omega)$$
$$f(x) \rightarrow f(x)\psi\left(\frac{x}{\varepsilon}\right)$$

is bounded, invertible and bicontinuous.

Remark 2.3. Recall that $\psi_{1,\theta}$ is exponentially decreasing as $(\exp(\theta y_n))$ when y_n goes to $-\infty$. The first part of (2.7) tells us that ψ has the same behavior and that the difference between ψ and a multiple of $\psi_{1,\theta}$ is decreasing with a faster exponential decay. Another way of writing (2.7) is to state that

$$\lim_{y_n \to -\infty} e^{-\tau y_n} \left| \nabla \left(\frac{\psi(y)}{\psi_{1,\theta}(y)} \right) \right| = 0.$$

Remark 2.4. The constants c_1 and c_2 appearing in (2.7) depend on the normalization of the cell eigenfunctions $\psi_{1,\theta}$ and ψ_2. We can choose this normalization such that $c_1 = c_2 = 1$. Indeed, denoting by $n_1 > 0$ and $n_2 > 0$ the positive constants to which N_1 and N_2 stabilize at infinity, multiplying $\psi_{1,\theta}$ by a constant K changes the constants $c_1 = 1+n_1$ and $c_2 = 1+n_2$ in new constants $(1+K^{-1}n_1)$ and $(1 + Kn_2)$. Taking $K = \sqrt{n_1/n_2}$ gives $(1 + K^{-1}n_1) = (1 + Kn_2)$ and this unique constant can be eliminated by multiplying it to the resulting ψ.

Similar results can be obtained for ψ_0, which is defined with the two periodic eigensolutions ψ_1 and ψ_2.

Corollary 2.5. *The function* ψ_0 *defined by* (2.2) *belongs to* $H^1(G)$*, is continuous through the interface* Γ*, and satisfies, for* $i = 1, 2$ (*with different eigenvalues* μ_i),

$$-\operatorname{div}(a_i \nabla \psi_0) + \Sigma_i \psi_0 = \mu_i \sigma_i \psi_0 \quad \text{in } G_i,$$

and there exist three positive constants $\tau > 0$, $c_1^0 > 1$ *and* $c_2^0 > 1$ *such that*

$$\lim_{y_n \to -\infty} e^{-\tau y_n} |\nabla(\psi_0(y) - c_1^0 \psi_1(y))| = 0,$$
$$\lim_{y_n \to \infty} e^{\tau y_n} |\nabla(\psi_0(y) - c_2^0 \psi_2(y))| = 0.$$

As a consequence, there exists a positive constant $C > 0$ *such that* $C^{-1} \le \psi_0(y) \le C$ *in* G*. The mapping*

$$T_0 : H^1_{\#,0}(\Omega) \rightarrow H^1_{\#,0}(\Omega)$$
$$f(x) \rightarrow f(x)\psi_0\left(\frac{x}{\varepsilon}\right)$$

is also a homeomorphism.

Remark 2.6. Note that we can always multiply ψ by an appropriate constant so that ψ/ψ_0 converges strongly to 1 in $L^p(\Omega_2)$ for any $1 \le p < \infty$. In the sequel, we will always assume that such choice was made.

The proof of Corollary 2.5 is quite standard and much easier than that of Proposition 2.2. Actually, the boundary layers $N_{1,0}$ and $N_{2,0}$, defined by (2.1), are solutions of self-adjoint problems with periodic coefficients. In such a case, the exponential stabilization to a constant of $N_{1,0}$, $N_{2,0}$, as well as the exponential decay of their gradients in the direction normal to the interface is a well known result (see e.g. [22], [23], [24]). However, the proof of Proposition 2.2 is delicate because the coefficients of problem (2.4) are exponentially decreasing in G_1. The main trick of the proof is to show that (2.4) is equivalent to a problem with periodic coefficients which is no longer self-adjoint. Therefore, we need to replace the classical results of [22], [23], [24] by a more general result of [27, 28] (see Theorem 2.7 below).

Proof of Proposition 2.2. First of all, equation (2.6) is just a matter of simple algebra. If ψ exists, by our smoothness assumption on the coefficients it belongs to $W^{1,\infty}(G)$, so the mapping T is a homeomorphism. Therefore, the only point to check carefully is the well-posedness of the boundary layer problems (2.4). The existence, uniqueness, and behavior at infinity of N_2 is classical (see e.g. [22], [23], [24]). Indeed, because of our smoothness assumption on the coefficients, the cell eigenfunction ψ_2 is continuous, so there exists a positive constant $C > 0$ such that $C \ge \psi_2(y) \ge C^{-1}$ for all $y \in G$. The boundary layer N_2 is thus uniquely defined in a Deny-Lions space as the solution of an uniformly elliptic boundary value problem posed on a semi-infinite strip. Furthermore, there exist two constants n_2 and $\tau > 0$ such that

$$\lim_{y_n \to +\infty} e^{\tau y_n}\left(|N_2(y) - n_2| + |\nabla N_2(y)|\right) = 0.$$

By the maximum principle, the constant n_2 must be positive since $\psi_{1,\theta}/\psi_2$ is positive on Γ. Finally we have $c_2 = 1 + n_2$.

Let us now turn to the case of N_1. We introduce a periodic function $\phi_{1,\theta}$ defined by

$$\phi_{1,\theta}(y) = \psi_{1,\theta}(y)e^{-\theta y_n}.$$

Because of our smoothness assumption on the coefficients, this function is continuous and there exists a positive constant $C > 0$ such that $C \ge \phi_{1,\theta}(y) \ge C^{-1}$ for all $y \in G$. We then rewrite equation (2.4) in G_1 as

$$\begin{cases} -\operatorname{div}(\phi_{1,\theta}^2 a_1(y)\nabla N_1) - 2\theta\phi_{1,\theta}^2 a_1^n(y)\cdot\nabla N_1 = 0 & \text{in } G_1, \\ N_1(\cdot, y_n), & \mathbb{T}^{n-1}\text{periodic}, \\ N_1(\cdot,0) = \dfrac{\psi_2}{\psi_{1,\theta}}(\cdot,0), \end{cases} \tag{2.9}$$

where a_1^n is the n^{th} column of the matrix a_1. The point is that (2.9) has purely periodic coefficients. By application of Theorem 2.7 below, there exists a unique solution of (2.9) with the required asymptotic behavior at infinity if we can show that

$$\bar{b} = \int_{\mathbb{T}^n} (-\operatorname{div}(\phi_{1,\theta}^2 a_1) - 2\theta \phi_{1,\theta}^2 a_1^n) p^* \, dy$$

satisfies $\bar{b} \cdot e_n \leq 0$, with p^* the first eigenfunction of the adjoint cell problem satisfying

$$(2.10) \qquad -\operatorname{div}(\phi_{1,\theta}^2 a_1(y) \nabla p^*) + 2\theta \operatorname{div}(\phi_{1,\theta}^2 a_1^n(y) p^*) = 0 \quad \text{in } \mathbb{T}^n.$$

Let us show that

$$(2.11) \qquad \bar{b} = \frac{d\mu_1}{d\theta}(\theta),$$

which is non-positive since $\theta \geq 0$ and $\theta \to \mu_1(\theta)$, is strictly concave by Lemma 2.1. To obtain (2.11) we rewrite (2.3) as

$$(2.12) \qquad -\operatorname{div}(a_1 \nabla \phi_{1,\theta}) - 2\theta a_1^n \cdot \nabla \phi_{1,\theta} + (\Sigma_1 - \theta \operatorname{div}(a_1^n) - \theta^2 a_1^{nn}) \phi_{1,\theta} = \mu_1(\theta) \sigma_1 \phi_{1,\theta} \quad \text{in } \mathbb{T}^n,$$

where a_1^{nn} is the n^{th} component of the vector a_1^n, or equivalently the (n,n)-entry of the matrix a_1. We introduce the adjoint equation of (2.12), which admits the same first eigenvalue $\mu_1(\theta)$,

$$(2.13) \qquad -\operatorname{div}(a_1 \nabla \phi_{1,\theta}^*) + 2\theta \operatorname{div}(a_1^n \phi_{1,\theta}^*) + (\Sigma_1 - \theta \operatorname{div}(a_1^n) - \theta^2 a_1^{nn}) \phi_{1,\theta}^* = \mu_1(\theta) \sigma_1 \phi_{1,\theta}^* \quad \text{in } \mathbb{T}^n.$$

We normalize the first eigenfunction $\phi_{1,\theta}^*$ by

$$\int_{\mathbb{T}^n} \sigma_1 \phi_{1,\theta} \phi_{1,\theta}^* \, dy = 1.$$

As a matter of simple algebra we have $\phi_{1,\theta}^* = \phi_{1,\theta} p^*$ with p^* the solution of (2.10). Since the first eigenvalue of (2.12) is simple, we can differentiate (2.12) with respect to θ. We write (2.12) in abstract form as $\mathcal{A}(\theta)\phi_{1,\theta} = 0$, where $\mathcal{A}(\theta)$ is an operator acting in $\mathbb{T}^n$. We obtain

$$\begin{aligned} \mathcal{A}(\theta) \frac{d\phi_{1,\theta}}{d\theta} &= -\frac{d\mathcal{A}(\theta)}{d\theta} \phi_{1,\theta} \\ &= 2a_1^n \cdot \nabla \phi_{1,\theta} + \left(\operatorname{div}(a_1^n) + 2\theta a_1^{nn} + \frac{d\mu_1}{d\theta}(\theta) \sigma_1 \right) \phi_{1,\theta} \quad \text{in } \mathbb{T}^n, \end{aligned}$$

where the right hand side must satisfy the Fredholm compatibility condition, namely must be orthogonal to $\phi^*_{1,\theta}$. This condition implies that

$$\frac{d\mu_1}{d\theta}(\theta) = -\int_{\mathbb{T}^n} (2a_1^n \cdot \nabla\phi_{1,\theta} + (\operatorname{div}(a_1^n) + 2\theta a_1^{nn})\phi_{1,\theta})\phi^*_{1,\theta}\, dy,$$

which is precisely the definition of $\bar{b}$. ◻

We now recall a result of [27, 28] concerning the existence and uniqueness of solutions of

$$\begin{cases} -\operatorname{div}(a\nabla v) + b \cdot \nabla v = 0 & \text{in } G_1, \\ v(y', 0) = v_0 & \text{on } \Gamma, \end{cases} \tag{2.14}$$

where a is a $\mathbb{T}^n$-periodic uniformly coercive tensor, b is a $\mathbb{T}^n$-periodic vector field, and v_0 is a given boundary data. As usual, we assume that these data have enough smoothness in order that the solutions of (2.14) are, at least locally, in $W^{1,\infty}(G_1)$. We consider here the semi-infinite strip G_1, but a similar result holds for G_2. We shall need the cell eigenvalue problem

$$-\operatorname{div}(a\nabla p) + b \cdot \nabla p = \lambda p \quad \text{in } \mathbb{T}^n, \tag{2.15}$$

and its adjoint

$$-\operatorname{div}(a\nabla p^*) - \operatorname{div}(bp^*) = \lambda p^* \quad \text{in } \mathbb{T}^n. \tag{2.16}$$

Clearly, the first eigenvalue of (2.15) is $\lambda = 0$ with the corresponding eigenfunction $p = 1$, and thus, there exists a positive first eigenfunction p^* of (2.16) which satisfies $-\operatorname{div}(a\nabla p^*) - \operatorname{div}(bp^*) = 0$. From now on p^* denotes this positive first eigenfunction.

Theorem 2.7 ([27, 28]). *Define*

$$\bar{b} = \int_{\mathbb{T}^n} (-\operatorname{div} a + b)p^*\, dy. \tag{2.17}$$

If $\bar{b} \cdot e_n \le 0$, problem (2.14) has a unique bounded solution. Moreover, there exist three constants $K \in \mathbb{R}$, $C > 0$ and $\tau > 0$ such that

$$|u(y) - K| \le Ce^{\tau y_n}.$$

If $\bar{b} \cdot e_n > 0$, for any $K \in \mathbb{R}$ there exist a bounded solution of (2.14) and two positive constant $C > 0$ and $\tau > 0$ such that

$$|u(y) - K| \le Ce^{\tau y_n}.$$

For each K, such a solution is unique, and there are no other bounded solutions.

Even though we are able to build a continuous function ψ in the infinite band G that stabilizes at infinity to $c_1\psi_{1,\theta}$ and $c_2\psi_2$, $c_1, c_2 > 1$, we cannot enforce the continuity of the flow $(a\nabla\psi)n$ normal to the interface Γ. In other words, ψ is not the solution of equation (2.6) in the whole strip G. We introduce a so-called discontinuity function α defined by

$$\begin{aligned}(2.18)\quad \alpha(y') &= \sum_{j=1}^{n} a_{1,nj}(y',0)\psi_{1,\theta}(y',0)(1+N_1(y',0))\frac{\partial(\psi_{1,\theta}(1+N_1))}{\partial y_j}(y',0) \\ &\quad - a_{2,nj}(y',0)\psi_2(y',0)(1+N_2(y',0))\frac{\partial(\psi_2(1+N_2))}{\partial y_j}(y',0) \\ &\qquad \text{for all } y' \in \Gamma.\end{aligned}$$

Due to our smoothness assumptions on the coefficients, $\alpha(y')$ is a function in $L^\infty(\Gamma)$. We also introduce

$$\begin{aligned}(2.19)\quad \alpha_0(y') &= \sum_{j=1}^{n} a_{1,nj}(y',0)\psi_1(y',0)(1+N_{1,0}(y',0))\frac{\partial(\psi_1(1+N_{1,0}))}{\partial y_j}(y',0) \\ &\quad - a_{2,nj}(y',0)\psi_2(y',0)(1+N_{2,0}(y',0))\frac{\partial(\psi_2(1+N_{2,0}))}{\partial y_j}(y',0) \\ &\qquad \text{for all } y' \in \Gamma.\end{aligned}$$

Clearly, $\alpha = \alpha_0$ if $\mu_1 = \mu_2$.

3. Main Results

The different situations referred to in Section 1 will depend on the discontinuity function α, defined by (2.18), through the following variational problem (when it admits a solution)

$$(3.1)\qquad \Lambda = \left(\inf_{u\in D^{1,2}(G),\ \int_\Gamma \alpha(y')u^2(y',0)\,dy'=-1} \int_G D(y)\nabla u\cdot\nabla u\,dy\right),$$

where $D(y) = a(y,y_n)\psi^2(y)$ for all $y \in G$, and $D^{1,2}(G)$ is a weighted Deny-Lions (or Beppo-Levi) space [18] defined by

$$(3.2)\quad D^{1,2}(G) = \Big\{\phi \in H^1_{\mathrm{loc}}(G) \mid y' \to \phi(y',y_n)\ \mathbb{T}^{n-1}\text{-periodic}, \\ \int_{G_1} e^{\theta y_n}|\nabla\phi|^2\,dy + \int_{G_2}|\nabla\phi|^2\,dy < +\infty\Big\}.$$

This problem is studied in detail in Section 6 below. At this point, remark simply that there is no admissible test functions in (3.1) if $\alpha \geq 0$ on Γ, and in such a case we set $\Lambda = +\infty$ as is usual in optimization.

The first result concerns the special case when the discontinuity function α defined by (2.18) is identically zero. We then obtain a generalization of Theorem 1.1.

Theorem 3.1. *Let λ_m^ε and ϕ_m^ε be the m^{th} eigenvalue and normalized eigenfunction of* (1.1)*, and assume that α defined by* (2.18) *is such that $\alpha \equiv 0$ on Γ. Let ψ be the function defined by* (2.5)*. Then*

$$\phi_m^\varepsilon(x) = u_m^\varepsilon(x)\psi\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \lambda_m^\varepsilon = \mu_2 + \varepsilon^2 \nu_m + o(\varepsilon^2),$$

where

- *if $\mu_1 > \mu_2$, then, up to a subsequence, u_m^ε converges strongly in $L^2(\Omega)$ to u_m, with $u_m = 0$ in Ω_1 and (ν_m, u_m) is the m^{th} eigenpair of the following homogenized problem*

$$\text{(3.3)} \qquad \begin{cases} -\operatorname{div}(\bar{D}_2 \nabla u) = \nu \bar{\sigma}_2 u & \text{in } \Omega_2, \\ \bar{D}_{2,nj} \dfrac{\partial u}{\partial x_j}(\cdot, 0) = u(\cdot, L) = 0; \end{cases}$$

- *if $\mu_1 = \mu_2$, then, up to a sub-sequence, u_m^ε converges weakly in $H^1_{\#,0}(\Omega)$ to u_m, and (ν_m, u_m) is the m^{th} eigenpair of the following homogenized problem*

$$\text{(3.4)} \qquad \begin{cases} -\operatorname{div}((\chi_1(x)\bar{D}_1 + \chi_2(x)\bar{D}_2)\nabla u), \\ \qquad\qquad = \nu(\chi_1(x)\bar{\sigma}_1 + \chi_2(x)\bar{\sigma}_2)u \quad \text{in } \Omega, \\ u(\cdot, -\ell) = 0 \quad \text{and} \quad u(\cdot, L) = 0. \end{cases}$$

In both cases, the homogenized coefficients are given, for $k = 1, 2$, by

$$\text{(3.5a)} \qquad (\bar{D}_k)_{ij} = \int_{\mathbb{T}^n} c_k^2 \psi_k^2(y) a_k(y)(\nabla \xi_i + e_i) \cdot (\nabla \xi_j + e_j)\, dy,$$

$$\text{(3.5b)} \qquad \bar{\sigma}_k = \int_{\mathbb{T}^n} \sigma_k(y) c_k^2 \psi_k^2(y)\, dy,$$

where c_k is the positive constant such that the function ψ is asymptotically equal to $c_k\psi_k$ at infinity in G_k (see Proposition 2.2) and the function ξ_i, for $1 \leq i \leq n$, is the solution of

$$\text{(3.6)} \qquad \begin{cases} -\operatorname{div}(\psi_k^2(y) a_k(y)(\nabla \xi_i + e_i)) = 0 & \text{in } \mathbb{T}^n, \\ y \to \xi_i(y), & \mathbb{T}^n\text{-periodic.} \end{cases}$$

Remark 3.2. In truth Proposition 2.2 claims that ψ is asymptotically equal to $c_1\psi_{1,\theta}$ at infinity in G_1. Nevertheless, the homogenized coefficient $\bar{D}_1$ is required only in the case $\mu_1 = \mu_2$ which corresponds to $\theta = 0$, and thus $\psi_{1,\theta} = \psi_1$. Therefore, in such a case it is true that ψ is asymptotically equal to $c_1\psi_1$ at infinity in G_1.

If the discontinuity function is not zero almost everywhere but $\Lambda = 1$, then the discontinuity is removable, and we obtain the following result.

Theorem 3.3. *Let λ_m^ε and ϕ_m^ε be the m^{th} eigenvalue and normalized eigenfunction of* (1.1). *Assume that the minimal value of problem* (3.1) *is precisely $\Lambda = 1$. Then the conclusions of Theorem 3.1 are also valid provided that ψ is replaced by $\psi^* = (u^*\psi)$, where u^* is given by Theorem 6.2. (Here the homogenized coefficients are still defined by* (3.5)*, but with constants $c_k > 0$ being such that ψ^* is asymptotically equal to $c_k\psi_k$ at infinity.)*

Let us now turn to the cases where the discontinuity is not removable. Our first result concerns the case when no localization occurs and a Dirichlet boundary condition appears at the interface.

Theorem 3.4. *Let λ_m^ε and ϕ_m^ε be the m^{th} eigenvalue and normalized eigenfunction of* (1.1). *Assume that either $\Lambda > 1$, or*

$$\alpha(y') \geq 0 \text{ a.e. on } \Gamma \quad \text{and} \quad \int_\Gamma \alpha \, dy' > 0.$$

Then,

$$\phi_m^\varepsilon(x) = u_m^\varepsilon(x)\psi\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \lambda_m^\varepsilon = \mu_2 + \varepsilon^2 \nu_m + o(\varepsilon^2),$$

where

- *if $\mu_1 = \mu_2$, then, up to a sub-sequence, u_m^ε converges weakly in $H^1_{\#,0}(\Omega)$ to u_m, and (ν_m, u_m) is the m^{th} eigencouple of the homogenized problem*

$$\begin{cases} -\operatorname{div}(\bar{D}_1 \nabla u) = \nu \bar{\sigma}_1 u & \text{in } \Omega_1, \\ -\operatorname{div}(\bar{D}_2 \nabla u) = \nu \bar{\sigma}_2 u & \text{in } \Omega_2, \\ u(\cdot, -\ell) = u(\cdot, 0) = u(\cdot, L) = 0; \end{cases} \tag{3.7}$$

- *if $\mu_1 > \mu_2$, then u_m^ε converges strongly to 0 in $L^2(\Omega_1)$, and, up to a subsequence, u_m^ε converges weakly in $H^1_{\#,0}(\Omega_2)$ to u_m, and (ν_m, u_m) is the m^{th} eigenpair of the homogenized problem*

$$\begin{cases} -\operatorname{div}(\bar{D}_2 \nabla u) = \nu \bar{\sigma}_2 u & \text{in } \Omega_2, \\ u(\cdot, 0) = u(\cdot, L) = 0. \end{cases} \tag{3.8}$$

In both cases, the homogenized coefficients are defined by formula (3.5) for each half domain.

Finally, in all other cases we obtain a localization phenomena.

Theorem 3.5. *Assume that the minimal value of problem (3.1) satisfies $\Lambda < 1$. The first eigenvalue λ_1^ε of (1.1) converges to a limit $0 < \lambda_1 < \min(\mu_1, \mu_2)$, and, for some $\tau > 0$,*

$$|\lambda_1^\varepsilon - \lambda_1| < C \exp\left(-\frac{\tau}{\varepsilon}\right),$$

whereas the first normalized eigenvector satisfies

$$\left\|\nabla \phi_1^\varepsilon(x) - \frac{1}{\sqrt{\varepsilon}} \nabla \left(\Psi\left(\frac{x}{\varepsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\varepsilon(x) - \frac{1}{\sqrt{\varepsilon}} \Psi\left(\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leq C \exp\left(-\frac{\tau}{\varepsilon}\right),$$

and $(\lambda_1, \Psi(y))$ is the first eigenpair of the eigenvalue problem

$$-\operatorname{div}(a(y, y_n)\nabla\Psi) + \Sigma(y, y_n)\Psi = \lambda_1 \sigma(y, y_n)\Psi \quad \text{in } G.$$

Furthermore, λ_1 is simple, while Ψ can be chosen positive and is exponentially decreasing away from the interface.

Remark 3.6. The criterion which selects the different limit cases is mainly the minimal value Λ of the auxiliary variational problem (3.1). In one dimension this criterion can be further explicited in terms of the value of α (a constant) at the interface (see [3]). The reason is that one can use ordinary differential techniques and Floquet theory in one dimension to build explicit solutions of (3.1). This is, of course, not possible in higher dimension. In particular, the refined one-dimensional analysis of [3] shows that any positive value of $\Lambda > 0$ can be achieved, and thus all limit behaviors described above are attainable. This is indeed confirmed by numerical simulations [3].

4. PROOFS IN ABSENCE OF LOCALIZATION

This section is devoted to the proofs of Theorems 3.1, 3.3 and 3.4. The strategy is to perform a change of unknowns (the so-called factorization principle) and then to prove the convergence of the spectrum by studying the convergence of the Green operator of a source problem. With the help of the particular solution ψ defined by (2.5), the eigenvalue problem (1.1) can be transformed into the following one, where the singular perturbation in front of the divergence term has disappeared. Proposition 4.1 gives the form of this new problem after some simple algebra.

Proposition 4.1. *Introducing* $u^\varepsilon(x) = \phi^\varepsilon(x)/\psi(x/\varepsilon)$, *the eigenvalue problem* (1.1) *is equivalent to*

$$(4.1)\qquad \begin{cases} -\operatorname{div}(D^\varepsilon \nabla u^\varepsilon) + \dfrac{1}{\varepsilon}\alpha\left(\dfrac{x'}{\varepsilon}\right) u^\varepsilon(x',0)\delta_{x_n=0} = \nu_\varepsilon B^\varepsilon u^\varepsilon \quad \text{in } \Omega, \\ u^\varepsilon \in H^1_{\#,0}(\Omega). \end{cases}$$

where $\alpha(y')$ *is the periodic function on* Γ *defined by* (2.18), *and with the notation*

$$D^\varepsilon(x) = a\left(\frac{x}{\varepsilon}, x_n\right)\psi^2\left(\frac{x}{\varepsilon}\right),$$
$$B^\varepsilon(x) = \sigma\left(\frac{x}{\varepsilon}, x_n\right)\psi^2\left(\frac{x}{\varepsilon}\right), \qquad \nu_\varepsilon = \frac{\lambda_\varepsilon - \mu_2}{\varepsilon^2}.$$

Note that the coefficients D^ε and B^ε are no longer periodic, but rather the superposition of periodic and exponential functions. Following a strategy already used in [4], [3], the asymptotic study of the eigenvalue problem (4.1) relies on the detailed homogenization, as ε tends to zero, of the following source problem (i.e., with given right hand side)

$$(4.2)\qquad \begin{cases} -\operatorname{div}(D^\varepsilon \nabla v_\varepsilon) + \dfrac{1}{\varepsilon}\alpha\left(\dfrac{x'}{\varepsilon}\right) v_\varepsilon(x',0)\delta_{x_n=0} = P^\varepsilon f_\varepsilon \quad \text{in } \Omega, \\ v_\varepsilon \in H^1_{\#,0}(\Omega), \end{cases}$$

with $P^\varepsilon(x) = \psi(x/\varepsilon)/\psi_0(x/\varepsilon)$, and with a right hand side f_ε which is a bounded sequence of $L^2(\Omega)$, weakly converging to a limit $f \in L^2(\Omega)$. We first obtain a priori estimates.

Proposition 4.2. *Suppose that either* $\alpha \geq 0$ *or* $\Lambda > 1$. *Then the solution* v_ε *of equation* (4.2) *satisfies*

$$(4.3)\qquad \|\nabla v_\varepsilon\|_{L^2(\Omega_2)} + \|e^{\theta x_n/\varepsilon}\nabla v_\varepsilon\|_{L^2(\Omega_1)} + \sqrt{\frac{\mu_1 - \mu_2}{\varepsilon}}\|P^\varepsilon v_\varepsilon\|_{L^2(\Omega_1)} \leq C\|f_\varepsilon\|_{L^2(\Omega)},$$

$$(4.4)\qquad \frac{1}{\varepsilon}\left|\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right) v_\varepsilon(x',0)^2\,dx\right| \leq C\|f_\varepsilon\|^2_{L^2(\Omega)},$$

where C *is a constant independent of* ε.

As a consequence, if $\Lambda > 1$, *or if* $\alpha \geq 0$ *and* $\int_\Gamma \alpha(y')\,dy' > 0$, *then*

- *if* $\mu_1 = \mu_2$, *up to a subsequence,* v_ε *converges weakly to a limit* v *in* $H^1_0(\Omega)$. *The limit satisfies* $v(x',0) = 0$ *and thus can be written as* $v = v_1 + v_2$ *with* $v_1 \in H^1_0(\Omega_1)$ *and* $v_2 \in H^1_0(\Omega_2)$.
- *if* $\mu_1 > \mu_2$, $P^\varepsilon v_\varepsilon$ *tends to zero in* $L^2(\Omega_1)$, *and up to a subsequence,* v_ε *converges weakly in* $H^1(\Omega_2)$ *to a limit* $v \in H^1_0(\Omega_2)$.

Alternatively, if $\alpha(y') = 0$ *almost everywhere on* Γ,

- *if* $\mu_1 = \mu_2$, *up to a subsequence,* v_ε *converges weakly in* $H_0^1(\Omega)$;
- *if* $\mu_1 > \mu_2$, $P^\varepsilon v_\varepsilon$ *tends to zero in* $L^2(\Omega_1)$, *and up to a subsequence,* v_ε *converges weakly in* $H^1(\Omega_2)$.

Admitting momentarily this proposition, let us turn to the proofs of our main results. We introduce a Green operator S_ε defined by

$$\begin{aligned} S_\varepsilon : L^2(\Omega) &\to L^2(\Omega) \\ f &\to w_\varepsilon = P^\varepsilon v_\varepsilon, \quad \text{with } v_\varepsilon \text{ being} \\ &\qquad \text{the unique solution in } H_0^1(\Omega) \text{ of equation (4.2) with r.h.s. } f. \end{aligned} \tag{4.5}$$

For all fixed $\varepsilon > 0$, S_ε is clearly a linear compact operator in $L^2(\Omega)$. We shall show the following result.

Proposition 4.3. *Let* f_ε *be a sequence weakly converging to a limit* f *in* $L^2(\Omega)$. *The sequence* $w_\varepsilon = P^\varepsilon v_\varepsilon = S_\varepsilon(f_\varepsilon)$ *converges strongly in* $L^2(\Omega)$ *to* w *defined by* $w = S(f)$.

(A.1) *If* $\alpha \equiv 0$ *and* $\mu_1 = \mu_2$, *then* S *is the following compact operator*

$$S : L^2(\Omega) \to L^2(\Omega)$$

$$f \to w, \text{ unique solution of } \begin{cases} -\operatorname{div}((\chi_1(x)\bar{D}_1 + \chi_2(x)\bar{D}_2)\nabla w) = f & \text{in } \Omega, \\ w(\cdot, -\ell) = 0 \quad \text{and} \quad w(\cdot, L) = 0, \\ y' \to w(y) \quad \mathbb{T}^{n-1}\text{-periodic.} \end{cases}$$

(A.2) *If* $\alpha \equiv 0$ *and* $\mu_1 > \mu_2$, *then* S *is the following compact operator*

$$S : L^2(\Omega) \to L^2(\Omega)$$

$$f \to w = 0 \text{ in } \Omega_1, \text{ and unique solution in } \Omega_2 \text{ of} \begin{cases} -\operatorname{div}(\bar{D}_2\nabla w) = f & \text{in } \Omega_2, \\ \bar{D}_{2,nj}\dfrac{\partial w}{\partial x_j}(\cdot, 0) = w(\cdot, L) = 0, \\ y' \to w(y) \quad \mathbb{T}^{n-1}\text{-periodic.} \end{cases}$$

(B.1) *If* $\alpha \geq 0$ *and* $\int_\Gamma \alpha\, dy' > 0$*, or* $\Lambda > 1$*, and* $\mu_1 = \mu_2$*, then* S *is the following compact operator*

$$S : L^2(\Omega) \to L^2(\Omega)$$

$$f \to w \text{ unique solution of} \begin{cases} -\operatorname{div}(\bar{D}_1 \nabla w) = f & \text{in } \Omega_1, \\ -\operatorname{div}(\bar{D}_2 \nabla w) = f & \text{in } \Omega_2, \\ w(\cdot, -\ell) = w(\cdot, 0) = w(\cdot, L) = 0, \\ y' \to w(y) \quad \mathbb{T}^{n-1}\text{-periodic.} \end{cases}$$

(B.2) *If* $\alpha \geq 0$ *and* $\displaystyle\int_\Gamma \alpha\, dy' > 0$*, or* $\Lambda > 1$*, and* $\mu_1 > \mu_2$*, then* S *is the following compact operator*

$$S : L^2(\Omega) \to L^2(\Omega)$$

$$f \to w = 0 \text{ in } \Omega_1, \quad \text{and unique solution in } \Omega_2 \text{ of} \begin{cases} -\operatorname{div}\left(\bar{D}_2 \nabla w\right) = f & \text{in } \Omega_2, \\ w(\cdot, 0) = w(\cdot, L) = 0, \\ y' \to w(y) \quad \mathbb{T}^{n-1}\text{-periodic.} \end{cases}$$

In all cases, $\bar{D}_1$ *and* $\bar{D}_2$ *are given by* (3.5).

Proof. In the cases (A.1) and (B.1) we have $\mu_1 = \mu_2$, so that $\psi = \psi_0$, i.e., $P^\varepsilon = 1$, and thus $w_\varepsilon = v_\varepsilon$. In these cases the diffusion coefficient of (4.2) stabilizes at infinity to coercive periodic coefficients (indeed, they are the superposition of periodic and exponentially decreasing functions of the type $\exp(-\varepsilon^{-1}|x_n|)$ which converges strongly to zero in any $L^p(\Omega)$ with $1 \leq p < +\infty$). The proof in Cases (A.1) and (B.1) are quite standard in homogenization theory, with the a priori estimates of Proposition 4.2. For example, using the method of the oscillating test function [13], [25], or that of two-scale convergence [1], [26], it is an easy exercise that we safely leave to the reader. Let us simply remark that the homogenization of (4.2) is completely obvious in Case (A.1), while in Case (B.1), Proposition 4.2 shows that $v_\varepsilon(\cdot, 0)$ converges to zero in $L^2(\Gamma)$. Let us now turn to the other two cases for which $w_\varepsilon \neq v_\varepsilon$.

Case (A.2). From Proposition 4.2 we know that w_ε converges strongly to 0 in $L^2(\Omega_1)$. On the other hand, see Remark 2.6, P^ε converges strongly to 1 in $L^p(\Omega_2)$ for any $1 \leq p < +\infty$. Therefore, it is enough to prove the weak convergence of v_ε in $H^1(\Omega_2)$ to obtain the desired result. Testing variationally equation (4.2) against a test function $\phi_\varepsilon \in H^1_{\#,0}(\Omega)$, we obtain

$$\int_\Omega a\left(\frac{x}{\varepsilon}, x_n\right) \psi^2\left(\frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla \phi_\varepsilon \, dx = \int_\Omega P^\varepsilon f_\varepsilon \phi_\varepsilon \, dx. \tag{4.6}$$

Note that for any bounded sequence $\phi_\varepsilon \in W^{1,\infty}(\Omega)$,

$$\left| \int_{\Omega_1} a\left(\frac{x}{\varepsilon}, x_n\right)\psi^2\left(\frac{x}{\varepsilon}\right)\nabla v_\varepsilon \cdot \nabla\phi_\varepsilon \, dx \right| \leq C \left\| \exp\left(\theta \frac{x_n}{\varepsilon}\right) \nabla v_\varepsilon \right\|_{L^2(\Omega_1)} \left\| \exp\left(\theta \frac{x_n}{\varepsilon}\right) \nabla \phi_\varepsilon \right\|_{L^2(\Omega_1)} \to 0,$$

since $\| \exp(\theta x_n/\varepsilon)\nabla v_\varepsilon\|_{L^2(\Omega_1)}$ is bounded, thanks to Proposition 4.2. Of course, $\int_{\Omega_1} P^\varepsilon f_\varepsilon \phi_\varepsilon \, dx$ goes to zero. Consequently, for such sequences ϕ_ε uniformly bounded in $W^{1,\infty}(\Omega)$, identity (4.6) writes

$$\int_{\Omega_2} D^\varepsilon \nabla v_\varepsilon \cdot \nabla \phi_\varepsilon \, dx = \int_{\Omega_2} P^\varepsilon f_\varepsilon \phi_\varepsilon + o(1). \tag{4.7}$$

Since the test functions in the homogenization method are of the type $\phi_\varepsilon(x) = \phi_0(x) + \varepsilon\phi_1(x, x/\varepsilon)$ with smooth functions ϕ_0 and ϕ_1, they are uniformly bounded in $W^{1,\infty}(\Omega)$ and one can use (4.7) to pass to the limit. Classical arguments of homogenization theory allow us to conclude.

Case B.2. As in case (A.2), it is enough to study the weak convergence of v_ε in $H^1(\Omega_2)$ to obtain the desired result. Proposition 4.2 shows that $v_\varepsilon(y', 0)$ converges to zero in $L^2(\Gamma)$. Testing variationally equation (4.2) against ϕ_ε, where ϕ_ε is a test function in $W^{1,\infty}(\Omega) \cap H^1_{0,\#}(\Omega_2)$, we obtain again equation (4.7), and conclude using similar arguments to that of Case A.2. □

We are now able to conclude the proof of Theorem 3.1 and Theorem 3.4.

Proof of Theorem 3.1 and Theorem 3.4. Let us first remark that, since S is compact, Proposition 4.3 implies that the sequence of operators S_ε, defined by (4.5), uniformly converges to the limit operator S. The asymptotic analysis of the eigenvalue problem (4.1) is truly governed by the convergence of T_ε defined by

$$T_\varepsilon : L^2(\Omega) \to L^2(\Omega)$$
$$f \to S_\varepsilon\left(\frac{B^\varepsilon}{(P^\varepsilon)^2} f\right),$$

since the eigenvalues of T_ε are the inverse of that of (4.1). Remark that

$$\frac{B^\varepsilon(x)}{(P^\varepsilon(x))^2} = \sigma\left(\frac{x}{\varepsilon}, x_n\right)\left(\psi_0\left(\frac{x}{\varepsilon}\right)\right)^2,$$

which is a superposition of periodic functions and exponentially decreasing ones. Thus, in Ω_i, for $i = 1, 2$, it converges to a positive constant which is the average on the unit torus of the periodic function $\sigma_i \psi_i^2$. Denoting by $\bar{\sigma}(x_n)$ the weak limit of $B^\varepsilon/(P^\varepsilon)^2$, we define the limit operator T by

$$T : L^2(\Omega) \to L^2(\Omega)$$
$$f \to S(\bar{\sigma} f).$$

The sequence T_ε does not uniformly converge to T, but the sequence T_ε is nevertheless collectively compact, in the sense that

$$\forall f \in L^2(\Omega) \text{ such that } \lim_{\varepsilon\to 0} \|T_\varepsilon(f) - T(f)\|_{L^2(\Omega)} = 0,$$
$$\text{the set } \{T_\varepsilon(f) : \|f\|_{L^2(\Omega)} \le 1, \varepsilon \ge 0\} \text{ is sequentially compact.}$$

Theorems 3.1 and 3.4 are then consequences of a classical result in operator theory (see e.g. [7], [17], or Chapter 11 in [20]). □

Proof of Theorem 3.3. Let $u^*(y)$ be the unique positive minimizer to problem (3.1). Since $\Lambda = 1$, it is a positive solution of

$$-\operatorname{div}(D\nabla u^*) + \delta_{y_n=0}\alpha(y')u^* = 0 \quad \text{in } G.$$

Therefore, the function $\psi^*(y) = u^*(y)\psi(y)$ is positive and satisfies

$$\begin{aligned} -\operatorname{div}(a\nabla\psi^*) &+ \Sigma\psi^* \\ &= \frac{1}{\psi}(-\operatorname{div}(D\nabla u^*) + \delta_{y_n=0}\alpha(y')u^*) + \mu_2\sigma u^*\psi = \mu_2\sigma\psi^* \quad \text{in } G. \end{aligned}$$

The difference between ψ^* and ψ is that ψ^* is a solution in the entire strip G. In other words, there is no discontinuity function for ψ^*, i.e., $\alpha^* \equiv 0$ on the interface Γ. Consequently, by the new change of variables $u^{*\varepsilon}(x) = \phi^\varepsilon(x)/\psi^*(x/\varepsilon)$, the eigenvalue problem (1.1) is equivalent to

$$\begin{cases} -\operatorname{div}(D^{*\varepsilon}\nabla u^{*\varepsilon}) = \nu_\varepsilon B^{*\varepsilon}u^{*\varepsilon} & \text{in } \Omega, \\ u^{*\varepsilon} \in H^1_{\#,0}(\Omega), \end{cases}$$

with the notation

$$D^{*\varepsilon}(x) = a\left(\frac{x}{\varepsilon}, x_n\right)\left(\psi^*\left(\frac{x}{\varepsilon}\right)\right)^2,$$
$$B^{*\varepsilon}(x) = \sigma\left(\frac{x}{\varepsilon}, x_n\right)\left(\psi^*\left(\frac{x}{\varepsilon}\right)\right)^2, \quad \nu_\varepsilon = \frac{\lambda_\varepsilon - \mu_2}{\varepsilon^2}.$$

This eigenvalue problem is equivalent to (4.1) with $\alpha \equiv 0$, and its homogenization is easy according to Proposition 4.3. □

Proof of Proposition 4.2. Multiplying equation (4.2) by v_ε and integrating by parts, we obtain

$$\int_\Omega D^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx + \frac{1}{\varepsilon} \int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right) v_\varepsilon(x',0)^2 \, dx' = \int_\Omega P^\varepsilon f_\varepsilon v_\varepsilon \, dx. \tag{4.8}$$

Introducing $w_\varepsilon = P^\varepsilon v_\varepsilon$, the right-hand-side of equation (4.8) is bounded by $\|f_\varepsilon\|_{L^2(\Omega)}\|w_\varepsilon\|_{L^2(\Omega)}$. Thus, we deduce from (4.8) that

- if $\alpha \geq 0$, both left hand side terms are positive, and a being coercive and because of the bounds (2.8) for ψ we obtain

$$||\nabla v_\varepsilon||^2_{L^2(\Omega_2)} + ||e^{\theta x_n/\varepsilon} \nabla v_\varepsilon||^2_{L^2(\Omega_1)} \leq C\|f_\varepsilon\|_{L^2(\Omega)} \|w_\varepsilon\|_{L^2(\Omega)}; \tag{4.9}$$

- if $\Lambda > 1$, then from Lemma 6.4 we deduce that

$$\begin{aligned}\frac{\Lambda - 1}{\Lambda} &\int_\Omega D^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \\ &\leq \int_\Omega D^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx + \frac{1}{\varepsilon}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right) v_\varepsilon^2(x',0)\, dx',\end{aligned}$$

  and therefore inequality (4.9) is also satisfied.

To conclude we also need to estimate w_ε. For this aim, we write the equation satisfied by $w_\varepsilon = P^\varepsilon v_\varepsilon$. A computation similar to that of Proposition 2.2 shows that problem (4.2) is equivalent to

$$\begin{cases} -\operatorname{div}(D_0^\varepsilon \nabla w_\varepsilon) + \dfrac{\mu_1 - \mu_2}{\varepsilon^2} B_0^\varepsilon \chi_{\Omega_1}(x) w_\varepsilon \\ \qquad\qquad + \dfrac{1}{\varepsilon}\alpha_0\left(\dfrac{x'}{\varepsilon}\right) w_\varepsilon(x',0)\delta_{x_n=0} = f_\varepsilon \quad \text{in } \Omega, \\ w_\varepsilon \in H^1_{\#,0}(\Omega), \end{cases} \tag{4.10}$$

with

$$D_0^\varepsilon(x) = a\left(\frac{x}{\varepsilon}, x_n\right)\psi_0^2\left(\frac{x}{\varepsilon}\right), \quad B_0^\varepsilon(x) = \sigma\left(\frac{x}{\varepsilon}, x_n\right)\psi_0^2\left(\frac{x}{\varepsilon}\right).$$

By an integration by parts of (4.10) tested against w_ε we obtain

$$\begin{aligned}\int_\Omega D_0^\varepsilon \nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx &+ \frac{\mu_1 - \mu_2}{\varepsilon^2}\int_{\Omega_1} B_0^\varepsilon w_\varepsilon^2 \, dx \\ &+ \frac{1}{\varepsilon}\int_\Gamma \alpha_0\left(\frac{x'}{\varepsilon}\right) w_\varepsilon(x',0)^2\, dx' = \int_\Omega f_\varepsilon w_\varepsilon \, dx.\end{aligned} \tag{4.11}$$

Let us next show that

$$\begin{aligned}\|\nabla v_\varepsilon\|_{L^2(\Omega_2)} &\leq C\,||f_\varepsilon||_{L^2(\Omega)}, \\ \sqrt{\mu_1 - \mu_2}\,\|w_\varepsilon\|_{L^2(\Omega_1)} &\leq C\varepsilon^{1/2}\|f_\varepsilon\|_{L^2(\Omega)}.\end{aligned} \tag{4.12}$$

Note that since $C^{-1} v_\varepsilon(x',0) \le w_\varepsilon(x',0) \le C v_\varepsilon(x',0)$, where C is a positive constant which does not depend on ε, we have the following bound

$$\int_\Gamma w_\varepsilon(x',0)^2\, dx' \le C\|\nabla v_\varepsilon\|^2_{L^2(\Omega_2)} \le C\|f_\varepsilon\|_{L^2(\Omega)}\,\|w_\varepsilon\|_{L^2(\Omega)}.$$

Consequently, the identity (4.11) yields

$$\begin{aligned}(\mu_1-\mu_2)\|w_\varepsilon\|^2_{L^2(\Omega_1)} &\le C\varepsilon\|f_\varepsilon\|_{L^2(\Omega)}\,\|w_\varepsilon\|_{L^2(\Omega)}\\ &\le C\varepsilon\|f_\varepsilon\|_{L^2(\Omega)}\big(\|w_\varepsilon\|_{L^2(\Omega_1)}+\|w_\varepsilon\|_{L^2(\Omega_2)}\big),\end{aligned}$$

and the r.h.s. in (4.9) can be estimated as follows

$$\|f_\varepsilon\|_{L^2(\Omega)}\,\|w_\varepsilon\|_{L^2(\Omega)} \le C\|f_\varepsilon\|_{L^2(\Omega)}\,\|w_\varepsilon\|_{L^2(\Omega_2)} \le C\|f_\varepsilon\|_{L^2(\Omega)}\,\|v_\varepsilon\|_{L^2(\Omega_2)},$$

which implies (4.12). Finally, combining (4.9) and (4.12) we obtain (4.3). Estimate (4.4) is then a consequence of (4.8).

Let us now turn to the consequences of these estimates.

If $\mu_1 = \mu_2$, then $\theta = 0$, $w_\varepsilon = v_\varepsilon$ and estimate (4.3) shows that w_ε is bounded in $H^1_0(\Omega)$. This yields that, up to a subsequence, v_ε converges strongly to a limit v in $L^2(\Gamma)$. Thus

$$\lim_{\varepsilon\to 0}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right) v_\varepsilon(x',0)^2\, dx = \int_\Gamma \alpha(y')\, dy' \int_\Gamma v(x',0)^2\, dx.$$

Note that $\int_\Gamma \alpha(y')\, dy' \ne 0$, either by assumption, or because $\Lambda \ne 0$ implies $\int_\Gamma \alpha(y')\, dy' \ne 0$ according to Theorem 6.2. Passing to the limit in estimate (4.4) then proves that $v(x',0) = 0$.

If $\mu_1 \ne \mu_2$, then estimate (4.3) shows that w_ε converges to zero in $L^2(\Omega_1)$. Furthermore, $\|\nabla v_\varepsilon\|_{L^2(\Omega_2)}$ is bounded. Thus, up to a subsequence, v_ε converges to a limit in $H^1(\Omega_2)$, and therefore strongly in $L^2(\Gamma)$. The argument with (4.4), already used in the case $\mu_1 = \mu_2$, proves here also that $v(x',0) = 0$. □

5. Proofs for the Localization Phenomenon

This section is devoted to the proof of Theorem 3.5.

Proposition 5.1. *If the minimal value Λ of problem (3.1) is smaller than 1, i.e., $\Lambda < 1$, then the first eigenvalue λ^1_ε of problem (1.1) is decreasing as ε goes to zero, and satisfies*

$$\lim_{\varepsilon\to 0}\lambda^1_\varepsilon < \mu_2 = \min(\mu_1,\mu_2),$$

where μ_1 and μ_2 are the periodic cell eigenvalues defined in (1.3).

Proof. Problem (1.1) is self-adjoint: its first eigenvalue is given by

$$\lambda_\varepsilon^1 = \min_{\phi \in H^1_{\#,0}(\Omega),\ \phi \neq 0} \frac{\varepsilon^2 \int_\Omega a\left(\frac{x}{\varepsilon}, x_n\right) \nabla\phi \cdot \nabla\phi \, dx + \int_\Omega \Sigma\left(\frac{x}{\varepsilon}, x_n\right) \phi^2 \, dx}{\int_\Omega \sigma\left(\frac{x}{\varepsilon}, x_n\right) \phi^2 \, dx},$$

which, thanks to Proposition 4.1, is equivalent to

$$\lambda_\varepsilon^1 = \mu_2 + \min_{\phi \in H^1_{\#,0}(\Omega),\ \phi \neq 0} \frac{\varepsilon^2 \int_\Omega D^\varepsilon \nabla\phi \cdot \nabla\phi \, dx + \varepsilon \int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right) \phi(x',0)^2 \, dx'}{\int_\Omega B^\varepsilon \phi^2 \, dx}. \tag{5.1}$$

Since the first eigenvalue of (1.1) is simple, the minimization problem in the right hand side of (5.1) admits a minimizer φ, unique up to a multiplicative constant (chosen in such a way that φ is positive and normalized). Furthermore, φ must be periodic of period ε in all coordinate directions tangential to the interface. Indeed, because of the periodicity of the coefficients, the function $\tilde{\varphi}(x', x_n) = \varphi(x' + i\varepsilon, x_n)$ is also a positive and normalized minimizer, for all $i \in \mathbb{N}^{n-1}$. By uniqueness of the minimizer, it must be equal to $\varphi(x', x_n)$. Thus, by periodicity the integrals in (5.1) reduce to a single band $(\varepsilon\mathbb{T}^{n-1}) \times]-\ell, L[$. By the change of variables $y = \varepsilon^{-1}x$, (5.1) is thus equivalent to

$$\lambda_\varepsilon^1 = \mu_2 + \min_{\phi \in H^1_{\#,0}(G_\varepsilon),\ \phi \neq 0} \frac{\int_{G_\varepsilon} D(y)\nabla\phi(y) \cdot \nabla\phi(y) \, dy + \int_\Gamma \alpha(y')\phi(y',0)^2 \, dy'}{\int_{G_\varepsilon} B(y)\phi(y)^2 \, dy}, \tag{5.2}$$

where $G_\varepsilon = \mathbb{T}^{n-1} \times]-\ell\varepsilon^{-1}, L\varepsilon^{-1}[$, and $D(y) = a(y, y_n)\psi(y)^2$, $B(y) = \sigma(y, y_n)\psi(y)^2$. By virtue of Theorem 6.2, there exist $u \in D^{1,2}(G)$ such that

$$\int_G D(y)\nabla u \cdot \nabla u \, dy \leq \frac{1+\Lambda}{2} < 1 \quad \text{and} \quad \int_\Gamma \alpha(y')u^2(y',0)\, dy = -1$$

(if $\Lambda > 0$, we take u as the unique positive minimizer of the auxiliary problem (3.1), while, if $\Lambda = 0$, u is chosen as one element of the minimizing sequence built in the proof of Theorem 6.2). Furthermore, away from the interface Γ, ∇u decays exponentially to zero while u stabilizes exponentially to a constant. Let us consider the function $z_\varepsilon(y) = u(y)C_\varepsilon(y_n)$, where $C_\varepsilon(y_n)$ is a cut-off function defined by

$$C_\varepsilon(y_n) = \begin{cases} 0 & \text{for } y_n < -\varepsilon^{-1}\ell, \\ \eta(y_n + \varepsilon^{-1}\ell) & \text{for } -\varepsilon^{-1}\ell \le y_n < \eta^{-1} - \varepsilon^{-1}\ell, \\ 1 & \text{for } \eta^{-1} - \varepsilon^{-1}\ell \le y_n < -\eta^{-1} + \varepsilon^{-1}L, \\ \eta(\varepsilon^{-1}L - y_n) & \text{for } \varepsilon^{-1}L - \eta^{-1} \le y_n < \varepsilon^{-1}L, \\ 0 & \text{for } \varepsilon^{-1}L \le y_n. \end{cases}$$

The value of the (small) constant $\eta > 0$ will be chosen later. Then, $z_\varepsilon \in H^1_{\#,0}(G_\varepsilon)$, and

$$\int_{G_\varepsilon} D(y)\nabla z_\varepsilon \cdot \nabla z_\varepsilon \, dy \le \frac{1+\Lambda}{2} + \int_{B_\varepsilon} D(y)\nabla z_\varepsilon \cdot \nabla z_\varepsilon \, dy,$$

where $B_\varepsilon = G_\varepsilon \cap ([-\varepsilon^{-1}\ell, \eta^{-1} - \varepsilon^{-1}\ell] \cup [\varepsilon^{-1}L - \eta^{-1}, \varepsilon^{-1}L])$. Using the Cauchy-Schwarz inequality we obtain

$$\int_{B_\varepsilon} D(y)\nabla z_\varepsilon \cdot \nabla z_\varepsilon \, dy \le C(\|\nabla u\|^2_{L^2(B_\varepsilon)} + \eta^2 \|u\|^2_{L^2(B_\varepsilon)}).$$

Since u is uniformly bounded in G, we have $\|u\|^2_{L^2(B_\varepsilon)} \le C\eta^{-1}$, uniformly with respect to ε. Thus, for sufficiently small η we can make $\eta^2\|u\|^2_{L^2(B_\varepsilon)}$ as small as we want. Now, for a fixed η we can choose ε small enough such that $\|\nabla u\|_{L^2(B_\varepsilon)}$ is small. Consequently, for some η and some ε_0 we have

$$\int_{B_{\varepsilon_0}} D(y)\nabla z_{\varepsilon_0} \cdot \nabla z_{\varepsilon_0} \, dy \le \frac{1-\Lambda}{4}.$$

This implies that

$$\int_{G_{\varepsilon_0}} D(y)\nabla z_{\varepsilon_0} \cdot \nabla z_{\varepsilon_0} \, dy + \int_\Gamma \alpha(y') z_{\varepsilon_0}(y',0)^2 \, dy' \le \frac{\Lambda - 1}{4} < 0.$$

Consequently, plugging the test function z_{ε_0} in (5.2) yields

$$\lambda^1_{\varepsilon_0} < \mu_2.$$

To conclude, remark that, by inclusion of spaces, identity (5.2) implies that λ^1_ε is non increasing as ε goes to zero. □

We perform a change of unknowns for the first eigenvector of (1.1)

$$u_\varepsilon(x) = \frac{\phi^1_\varepsilon(x)}{\psi_0(x/\varepsilon)}.$$

The new unknown is a solution of an equation similar to (4.10)

$$(5.3)\quad \begin{cases} -\operatorname{div}(D_0^\varepsilon \nabla u_\varepsilon) + \dfrac{\mu_1-\mu_2}{\varepsilon^2} B_0^\varepsilon \chi_{\Omega_1}(x) u_\varepsilon + \dfrac{1}{\varepsilon}\alpha_0\left(\dfrac{x'}{\varepsilon}\right) u_\varepsilon \delta_{x_n=0} \\ \qquad\qquad = \dfrac{\lambda_\varepsilon^1 - \mu_2}{\varepsilon^2} B_0^\varepsilon u_\varepsilon \quad \text{in } \Omega, \\ u_\varepsilon \in H^1_{\#,0}(\Omega). \end{cases}$$

By the same argument as in the proof of Proposition 5.1, u_ε is periodic in the tangential coordinate directions, so we can reduce (5.3) to a single strip. Then, performing the change of variables $y = x/\varepsilon$, we obtain that $\bar{u}_\varepsilon(y) = u_\varepsilon(x)$ satisfies

$$(5.4)\quad \begin{cases} -\operatorname{div}_y(D_0(y)\nabla_y \bar{u}_\varepsilon) + (\mu_1-\mu_2)B_0(y)\chi_{y_n<0}(y)\bar{u}_\varepsilon \\ \qquad + \alpha_0(y)\bar{u}_\varepsilon(y',0)\delta_{y_n=0} = (\lambda_\varepsilon^1 - \mu_2)B_0(y)\bar{u}_\varepsilon \quad \text{in } G_\varepsilon, \\ \bar{u}_\varepsilon \in H^1_{\#,0}(G_\varepsilon). \end{cases}$$

Lemma 5.2. *Assume that $\Lambda < 1$. Then, the function $\bar{u}_\varepsilon$ (and its gradient) decays exponentially to zero away from the interface, uniformly with respect to ε.*

Proof. We rewrite (5.4) in $G_\varepsilon^2 = G_\varepsilon \cap \{y_n > 0\}$ as

$$(5.5)\quad \begin{cases} -\operatorname{div}_y(D_0(y)\nabla_y \bar{u}_\varepsilon) + (\mu_2 - \lambda_\varepsilon^1)B_0(y)\bar{u}_\varepsilon = 0 \quad \text{in } G_\varepsilon^2, \\ \bar{u}_\varepsilon(x', L\varepsilon^{-1}) = 0 \quad \text{and} \quad \bar{u}_\varepsilon(x',0) = g_\varepsilon, \end{cases}$$

for some trace condition g_ε. Since $\mu_2 - \lambda_\varepsilon^1 \geq C > 0$ by virtue of Proposition 5.1, we can prove that $\bar{u}_\varepsilon$ decays exponentially away from the interface $y_n = 0$ uniformly in ε. Indeed, for $k \in \mathbb{N}$, defining $G_\varepsilon^{2,k} = G_\varepsilon \cap \{y_n > k\}$ and $\Gamma_\varepsilon^{2,k} = G_\varepsilon \cap \{y_n = k\}$, $\bar{u}_\varepsilon$ is also a solution of

$$\begin{cases} -\operatorname{div}_y(D_0(y)\nabla_y \bar{u}_\varepsilon) + (\mu_2 - \lambda_\varepsilon^1)B_0(y)\bar{u}_\varepsilon = 0 & \text{in } G_\varepsilon^{2,k}, \\ \bar{u}_\varepsilon(x', L\varepsilon^{-1}) = 0 \quad \text{and} \quad \bar{u}_\varepsilon = \bar{u}_\varepsilon(x',k) & \text{on } \Gamma_\varepsilon^{2,k}, \end{cases}$$

and thus it satisfies the a priori estimate

$$\|\bar{u}_\varepsilon\|^2_{H^1(G_\varepsilon^{2,k})} \leq C\|\bar{u}_\varepsilon\|^2_{H^{1/2}(\Gamma_\varepsilon^{2,k})},$$

where the constant $C > 0$ depends on D_0, B_0, $(\mu_2 - \lim_{\varepsilon\to 0}\lambda_\varepsilon^1)$ but not on ε nor on k. Clearly, we also have

$$\|\bar{u}_\varepsilon\|^2_{H^{1/2}(\Gamma_\varepsilon^{2,k})} \leq \|\bar{u}_\varepsilon\|^2_{H^1(G_\varepsilon^{2,k-1}\setminus G_\varepsilon^{2,k})} \leq \|\bar{u}_\varepsilon\|^2_{H^1(G_\varepsilon^{2,k-1})} - \|\bar{u}_\varepsilon\|^2_{H^1(G_\varepsilon^{2,k})}.$$

Combining these two inequalities implies

$$\|\bar u_\varepsilon\|^2_{H^1(G^{2,k}_\varepsilon)} \le \frac{C}{C+1}\|\bar u_\varepsilon\|^2_{H^1(G^{2,k-1}_\varepsilon)}. \tag{5.6}$$

It is then a classical matter to deduce from (5.6) that there exist $\tau > 0$ and $C > 0$ (independent of ε) such that

$$\|\exp(\tau y_n)\bar u_\varepsilon\|_{H^1(G^2_\varepsilon)} \le C.$$

A similar argument works for $G^1_\varepsilon = G_\varepsilon \cap \{y_n < 0\}$. □

As a consequence of Lemma 5.2, the sequence $\bar u_\varepsilon$ (extended by zero at infinity) is precompact in $H^1_\#(G)$. Therefore, up to a subsequence, it converges to a limit $\bar u$ which satisfies

$$\begin{cases} -\operatorname{div}_y(D_0(y)\nabla_y \bar u) + (\mu_1-\mu_2)B_0(y)\chi_{y_n<0}\bar u \\ \qquad\qquad + \alpha_0(y)\bar u\delta_{y_n=0} = (\lim_{\varepsilon\to 0}\lambda^1_\varepsilon - \mu_2)B_0(y)\bar u \quad \text{in } G, \\ \bar u \in H^1_\#(G). \end{cases} \tag{5.7}$$

Lemma 5.3. *The solution $\bar u$ is the first eigenfunction of problem (5.7) which has a simple first eigenvalue. In particular, this implies that the entire sequence $\bar u_\varepsilon$ converges to $\bar u$.*

Proof. Let us denote by $(\bar\lambda - \mu_2)$ and $\bar v$ a generic eigenvalue and eigenfunction of (5.7). The usual factorization argument tells us that $\bar\phi = \psi_0\bar v$ is an eigenfunction for

$$\begin{cases} -\operatorname{div}_y(a(y)\nabla_y\bar\phi) + \Sigma(y)\bar\phi = \bar\lambda\sigma(y)\bar\phi \quad \text{in } G, \\ \bar\phi \in H^1_\#(G). \end{cases} \tag{5.8}$$

It is well known that the spectrum of equation (5.8) can be decomposed as the union of an essential spectrum and of a discrete spectrum, and that eigenfunctions corresponding to eigenvalues in the discrete spectrum are exponentially decaying to 0 at infinity. We also know that the minimum value of the essential spectrum is precisely equal to $\mu_2 = \min(\mu_1, \mu_2)$. Therefore, $\lim_{\varepsilon\to 0}\lambda^1_\varepsilon$ is an eigenvalue of (5.8) which belongs to its discrete spectrum. Thus, the first eigenvalue λ_1 of (5.8) is also in the discrete spectrum and its corresponding eigenvectors are exponentially decaying to 0 at infinity: it is then a classical matter to prove that λ_1 is simple and that its eigenvector can be chosen positive. Since $\bar u$ is a positive, exponentially decaying, solution of (5.7), we deduce that $\lambda_1 = \lim_{\varepsilon\to 0}\lambda^1_\varepsilon$ and $\psi_0\bar u$ is the first eigenfunction of (5.8). □

Proof of Theorem 3.5. Let us denote by Ψ the first normalized eigenfunction and by λ_1 the first eigenvalue of (5.8). From Lemma 5.3 we know that $\Psi = \psi_0 \bar{u}$. Let us consider the function

$$v_\varepsilon(x) = \Psi\left(\frac{x}{\varepsilon}\right) C(x_n),$$

where $C(x_n)$ is a smooth cut-off function such that $C(x_n) \equiv 1$ in a neighborhood of 0, and it has compact support in $]-\ell, +L[$. Because of the exponential decay of $\bar{u}$, we have

$$\left\|\nabla\left(\Psi\left(\frac{x}{\varepsilon}\right)\right) - \nabla v_\varepsilon(x)\right\|_{L^2(\Omega)} + \left\|\Psi\left(\frac{x}{\varepsilon}\right) - v_\varepsilon(x)\right\|_{L^2(\Omega)} \leq C \exp\left(-\frac{\tau}{\varepsilon}\right).$$

Thus, it remains to prove that

$$\|\nabla\phi_1^\varepsilon(x) - \nabla\tilde{v}_\varepsilon(x)\|_{L^2(\Omega)} + \|\phi_1^\varepsilon(x) - \tilde{v}_\varepsilon(x)\|_{L^2(\Omega)} \leq C \exp\left(-\frac{\tau}{\varepsilon}\right), \tag{5.9a}$$

$$|\lambda_1 - \lambda_1^\varepsilon| \leq C \exp\left(-\frac{\tau}{\varepsilon}\right), \tag{5.9b}$$

with $\tilde{v}_\varepsilon = v_\varepsilon / \|v_\varepsilon\|_{L^2(\Omega)}$ and $\|v_\varepsilon\|_{L^2(\Omega)} \approx \sqrt{\varepsilon}$. The variational formulations of (5.8) and (1.1) read respectively

$$\lambda_1 = \min_{\phi \in H^1_\#(G)\setminus\{0\}} \frac{\displaystyle\int_G a(y)\nabla_y\phi \cdot \nabla_y\phi\, dy + \int_G \Sigma(y)\phi^2\, dy}{\displaystyle\int_G \sigma(y)\phi^2\, dy} \tag{5.10}$$

$$\lambda_1^\varepsilon = \min_{\phi \in H^1_{\#,0}(G_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_{G_\varepsilon} a(y)\nabla_y\phi \cdot \nabla_y\phi\, dy + \int_{G_\varepsilon} \Sigma(y)\phi^2\, dy}{\displaystyle\int_{G_\varepsilon} \sigma(y)\phi^2\, dy} \tag{5.11}$$

We obviously have $\lambda_1 \leq \lambda_1^\varepsilon$. On the other hand, by substituting the test function $\phi(y) = v_\varepsilon(\varepsilon y)$ in (5.11) we get $\lambda_1^\varepsilon \leq \lambda_1 + C\exp(-\tau/\varepsilon)$ and the second inequality (5.9) follows. The first one can now be justified by standard spectral gap arguments. ◻

6. AUXILIARY VARIATIONAL PROBLEM

This section deals with the variational problem (3.1), namely

$$\Lambda = \left(\inf_{u \in D^{1,2}(G),\ \int_\Gamma \alpha(y')u^2(y',0)\,dy' = -1} \int_G D(y)\nabla u \cdot \nabla u\, dy\right),$$

where $D(y) = a(y, y_n)\psi^2(y)$ and $D^{1,2}(G)$ is the weighted Deny-Lions space defined by (3.2). One of the difficulties of this problem is that, when $\mu_1 \neq \mu_2$, the tensor $D(y)$ is exponentially decreasing as y_n tends to $-\infty$ (it is uniformly coercive for $y_n > 0$). Nevertheless, we show that this problem is well-posed if the discontinuity function α is not non-negative and has a non-zero average.

Remark 6.1. If $\alpha \geq 0$, there is clearly no admissible test function for (3.1) and, as is usual in optimization, we set $\Lambda = +\infty$. In such a case, either Theorem 3.1 or Theorem 3.4 apply. If $\int_\Gamma \alpha < 0$, then $\Lambda = 0$ and the minimum is realized by the constant function $(-\int_\Gamma \alpha)^{-1/2}$.

Theorem 6.2. *Assume that the set $M = \{y' \in \Gamma \mid \alpha(y') < 0\}$ is of non zero measure. Then, if $\int_\Gamma \alpha(y')\,dy' \neq 0$, there exists a unique positive minimizer u^* to problem (3.1). If $\int_\Gamma \alpha(y')\,dy' = 0$, then $\Lambda = 0$ and the minimum is not attained. When it exists, the minimizer u^* stabilizes to a constant at infinity, while its gradient decays exponentially to zero.*

Proof. Let us first consider the case $\int_\Gamma \alpha(y')\,dy' = 0$. Consider a smooth function v with compact support in G such that $B = \int_\Gamma \alpha v\,dy'$ is not zero, and define $A = \int_\Gamma \alpha v^2\,dy'$ and $0 < C = \int_G D\nabla v \cdot \nabla v\,dy < \infty$. Then for all $k > 0$, the function $v_k = (1/k)v - (k^2 + A)/(2kB)$ satisfies

$$\int_\Gamma \alpha(y')v_k^2(y',0)\,dy' = -1, \quad \text{and} \quad \int_G D\nabla v_k \cdot \nabla v_k\,dy = \frac{C}{k^2},$$

so that $0 \leq \Lambda \leq C/k^2$ for all k. This implies that the minimum is zero and is not attained.

Let us now turn to the case $\int_\Gamma \alpha(y')\,dy' \neq 0$ and prove that the infimum of (3.1) is attained in the space $D^{1,2}(G)$ defined by (3.2). Define a compact subset of G by $G_0 = \mathbb{T}^{n-1} \times (-1,1)$. Remark that, in G_0, the tensor $D(y)$ is uniformly coercive. Consider a minimizing sequence u_k. It satisfies $\int_{G_0} D(y)\nabla u_k \cdot \nabla u_k\,dy < C$ for some constant C. Because of the coercivity of D on G_0, this implies $\|\nabla u_k\|_{L^2(G_0)} < C$. The Poincaré-Wirtinger inequality

$$\left\| u_k - \int_\Gamma u_k(y',0)\,dy' \right\|_{L^2(G_0)} \leq c\|\nabla u_k\|_{L^2(G_0)}, \tag{6.1}$$

shows the boundedness of the sequence $u_k - \displaystyle\int_\Gamma u_k(y',0)\,dy'$ in $L^2(G_0)$. Remark that

$$\frac{1}{\int_\Gamma \alpha}\int_\Gamma \alpha u_k^2 = \frac{1}{\int_\Gamma \alpha}\int_\Gamma \alpha\left(u_k - \int_\Gamma u_k\right)^2 + \frac{2}{\int_\Gamma \alpha}\int_\Gamma \alpha\left\{\left(u_k - \int_\Gamma u_k\right)\left(\int_\Gamma u_k\right) + \left(\int_\Gamma u_k\right)^2\right\}.$$

Since $\int_\Gamma \alpha(y')u_k^2(y',0)\,dy' = -1$, this implies

$$\left(\int_\Gamma u_k(y',0)\,dy'\right)^2 \le C\left(1 + \left|\int_\Gamma u_k(y',0)\,dy'\right|\right),$$

and consequently $|\int_\Gamma u_k(y',0)\,dy'|$ and in turn $\|u_k\|_{L^2(G_0)}$ are bounded. Thus, the sequence u_k is bounded in $H^1(G_0)$ and, up to a subsequence, u_k converges to a limit u weakly in $H^1(G_0)$ and strongly in $L^2(\Gamma)$. In particular this implies that the limit still satisfies the constraint $\int_\Gamma \alpha(y')u^2(y',0)\,dy' = -1$. Actually, the sequence u_k converges weakly to u in $D^{1,2}(G)$, and the weak lower semicontinuity of the quadratic functional $\int_G D\nabla u\cdot\nabla u\,dy$ yields that u is a minimizer. Remark that, if u is a minimizer, then $|u|$ is also a minimizer, so we can assume from now on that u is non-negative. Let us show that there exists a unique positive minimizer u^*. The Euler-Lagrange equation for a non-negative minimizer u^* of (3.1) is

$$-\operatorname{div}(D\nabla u^*) + \Lambda\alpha(y')\delta_{y_n=0}u^* = 0 \quad \text{in } G.$$

Applying Lemma 6.3 shows that u^* is indeed positive in G. On the same token, u^* is also bounded by a posivite constant K in a neighbourhood G_0 of the interface Γ. Define the truncation $u^K = \min(u^*, K)$ which is well defined in $D^{1,2}(G)$, and satisfies $u^K = u^*$ on Γ. Furthermore, $\nabla u^K = \nabla u^*$ where $u^K = u^*$, and $\nabla u^K = 0$ where $u^* > K$. Thus, we have

$$\Lambda = \int_G D\nabla u^*\cdot\nabla u^*\,dy \ge \int_G D\nabla u^K\cdot\nabla u^K\,dy$$

and the inequality is strict if $u^K \ne u^*$, which contradicts the fact that u^* is a minimizer. Therefore, u^* is uniformly bounded in G. Eventually, arguing as in the proof of Proposition 2.2, we may invoke Theorem 2.7 to obtain the uniqueness of the bounded positive solution of the Euler-Lagrange equation (as well as its desired behavior at infinity) and thus of the positive minimizer u^* of (3.1). □

Lemma 6.3. *There is a strong maximum principle for the equation*

$$-\operatorname{div}(D\nabla u) + \alpha(y')\delta_{y_n=0}u = 0 \quad \text{in } G,$$

i.e., if a nonnegative solution u is strictly positive at least at one point of G, then this solution is strictly positive everywhere in G.

Proof. We assume for definiteness that $u(y_0) > 0$ for some $y_0 \in \bar{G}_1$. Then, by the standard strong maximum principle, $u(y) > 0$ for any $y \in G_1$. The same holds true for G_2. It remains to prove that u is also positive on the interface Γ. Let v_1 and v_2 be the unique solutions of the following problems:

$$-\operatorname{div}(D\nabla v_1) = 0 \text{ in } \mathbb{T}^{n-1}\times(-1,0),\ v_1(y',-1) = 1+\beta,\ v_1(y',0) = \beta,$$
$$-\operatorname{div}(D\nabla v_2) = 0 \text{ in } \mathbb{T}^{n-1}\times(0,1),\ v_1(y',0) = \beta,\ v_1(y',1) = 0.$$

with $\beta > 0$. By the maximum principle, we have

$$-D_{1j}(y',0)\frac{\partial}{\partial y_j}v_1(y',0) = -D_{11}(y',0)\frac{\partial}{\partial y_1}v_1(y',0) > 0$$

for any $y' \in \Gamma$. By uniform continuity arguments this implies

$$-D_{11}(y',0)\frac{\partial}{\partial y_1}v_1(y',0) \geq c > 0.$$

It is easy to see that the gradient of v_1 does not depend on β. It is also clear that

$$0 < -D_{11}(y',0)\frac{\partial}{\partial y_1}v_2(y',0) \leq c_1\beta.$$

The function

$$v = \begin{cases} v_1, & -1 \leq y_n \leq 0, \\ v_2, & 0 \leq y_n \leq 1, \end{cases}$$

is continuous and satisfies the equation

$$-\operatorname{div}(D\nabla v) = D_{11}(y',0)\frac{\partial}{\partial y_1}v_1(\cdot,0) - D_{11}(y',0)\frac{\partial}{\partial y_1}v_2(\cdot,0)\delta(y_n). \tag{6.2}$$

Denoting

$$\kappa(y') = -\beta^{-1}\left(D_{11}(y',0)\frac{\partial v_1}{\partial y_1}(y',0) - D_{11}(y',0)\frac{\partial v_2}{\partial y_1}(y',0)\right),$$

since $v(y',0) = \beta$ on Γ, equation (6.2) can be rewritten as follows

$$-\operatorname{div}(D\nabla v) + \kappa(y')\delta_{y_n=0}v = 0.$$

For sufficiently small β we get

$$\kappa(y') \geq \frac{c}{2\beta},$$

which gives in turn $\kappa(y') \geq \alpha(y')$. Multiplying, if necessary, v by an appropriate positive constant γ, we obtain $u(y',-1) - \gamma v(y',-1) > 0$ for all $y' \in \mathbb{T}^{n-1}$. Substituting in the equation gives

$$\begin{aligned} &-\operatorname{div}(D\nabla(u-\gamma v)) + \alpha\delta_{y_n=0}(u-\gamma v) = \gamma(\kappa-\alpha)v \quad \text{in } \mathbb{T}^{n-1}\times(-1,+1), \\ &(u-\gamma v)(y',-1) \geq 0, \quad (u-\gamma v)(y',1) \geq 0, \end{aligned}$$

and, by the maximum principle, $u(y',0) \geq \gamma v(y',0) = \gamma\beta$. The positivity of u in G_2 now follows from the standard strong maximum principle. ◻

Lemma 6.4. *If the minimal value of problem* (3.1) *is such that* $\Lambda > 1$*, then there exists a constant* $C_\varepsilon > 0$ *such that, for all* $\phi \in H^1_{\#,0}(\Omega)$*,*

$$\frac{1}{\varepsilon}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right)\phi(x',0)^2\,dx' + \frac{1}{\Lambda}\int_\Omega D^\varepsilon\nabla\phi\cdot\nabla\phi\,dx \geq C_\varepsilon\int_\Omega \phi^2\,dx.$$

Proof. Consider φ a function which realizes the minimum of

$$\min_{\phi\in H^1_{\#,0}(\Omega),\ \int_\Omega\phi^2=1}\frac{1}{\varepsilon}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right)\phi(x',0)^2 + \frac{1}{\Lambda}\int_\Omega D\left(\frac{x}{\varepsilon},x_n\right)\nabla\phi\cdot\nabla\phi.$$

Clearly $|\varphi|$ is also a minimizer so we can assume that φ is non negative. Thanks to the strong maximum principle introduced in Lemma 6.3, φ is positive and uniquely determined. Remark that φ must be $\varepsilon\mathbb{T}^{n-1}$-periodic. Indeed, because of the periodicity of the coefficients α and D, the function $\tilde\varphi(x',x_n) = \varphi(x'+i\varepsilon,x_n)$ is also a positive minimizer, for all $i\in\mathbb{N}^{n-1}$. Because of the uniqueness of such a minimizer, they must be equal. Consequently, after the change of variable $y = x/\varepsilon$ we obtain

$$\frac{1}{\varepsilon}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right)\varphi(x',0)^2\,dx' + \frac{1}{\Lambda}\int_\Omega D^\varepsilon\nabla\varphi\cdot\nabla\varphi\,dx$$
$$= \frac{\varepsilon^{n-2}}{\varepsilon^{n-1}}\left(\int_\Gamma \alpha(y')\tilde\varphi(y',0)^2\,dy' + \frac{1}{\Lambda}\int_{G_\varepsilon} D(y)\nabla\tilde\varphi\cdot\nabla\tilde\varphi\,dy\right),$$

where $\tilde\varphi(y) = \varphi(\varepsilon y)$ for all y. If we extend $\tilde\varphi$ by 0 on $G\setminus G_\varepsilon$, we obtain

$$\frac{1}{\varepsilon}\int_\Gamma \alpha\left(\frac{x'}{\varepsilon}\right)\varphi(x',0)^2\,dx' + \frac{1}{\Lambda}\int_\Omega D^\varepsilon\nabla\varphi\cdot\nabla\varphi\,dx$$
$$= \frac{1}{\varepsilon}\left(\int_\Gamma \alpha(y')\tilde\varphi(y',0)^2\,dy' + \frac{1}{\Lambda}\int_G D(y)\nabla\tilde\varphi\cdot\nabla\tilde\varphi\,dy\right) \geq C_\varepsilon > 0,$$

because $\tilde\varphi$, being equal to zero in $G\setminus G_\varepsilon$, is not the minimizer of problem (3.1). □

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Grégoire Allaire: *Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France*.
gregoire.allaire@polytechnique.fr

Yves Capdeboscq: *Department of Mathematics, Rutgers University, Piscataway, NJ 08854, U. S. A*.
ycrc@math.rutgers.edu

Andrey Piatnitski: *Narvik Institute of Technology, HiN, P.O.Box 385, 8505, Narvik, Norway; also: P.N.Lebedev Physical Institute RAS, Leninski prospect 53, Moscow 117333, Russia*.
andrey@sci.lebedev.ru

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