MASTER M2 MATHEMATICS OF MODELING
UNIVERSITE PIERRE ET MARIE CURIE - ECOLE POLYTECHNIQUE
Course of G. Allaire, "Hyperbolic systems of conservation laws"
January 18th, 2018 (3 hours)

Important: The evaluation procedure will pay attention to the quality of the dissertation and most particularly to the clarity and readability of the proposed argumentation. The number of points, attributed to each part, is a tentative distribution, reflecting the relative difficulties of each part.

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1 Isentropic gas dynamics (7 points)

Let us consider Euler equations of isentropic gas dynamics in Lagrangian coordinates and one space dimension

$$\left\{ \begin{array}{l} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = 0, \end{array} \right. \tag{1}$$

where τ is the specific volume, u the velocity and p the pressure given by a state equation, satisfying for any $\tau > 0$,

$$p(\tau) > 0 \quad p'(\tau) < 0 \quad p''(\tau) > 0. \tag{2}$$

1. Prove that system (1) is strictly hyperbolic for $\tau > 0$ and $u \in \mathbb{R}$.
2. Prove that the two characteristic fields are truly non-linear (TNL).
3. Find one Riemann invariant for each field.
4. Define the function
$$S(\tau, u) = \frac{1}{2}u^2 + E(\tau),$$
where E is a primitive of $-p(\tau)$, namely $E'(\tau) = -p(\tau)$. Prove that $S(\tau, u)$ is a convex function and that it is an entropy of system (1). Give explicitly the entropy flux (which is very simple).
5. For this question, one chooses the pressure law $p(\tau) = 1/(3\tau^3)$. Compute the profile of rarefaction waves for each field. Plot in the (τ, u) plane the rarefaction curve for each field, that is the set of right states (τ_R, u_R) which can be connected to a given left state (τ_L, u_L).
6. Briefly recall the structure of the solution to the Riemann problem for (1) when the right (τ_R, u_R) and left (τ_L, u_L) states are close enough.

2 Relaxed system (13 points)

Let us consider a **relaxed** version of (1) which turns out to be useful for numerical purposes when solving the Riemann problem. For a small parameter $\epsilon > 0$, it is a system of 3 equations with a source term

$$\left\{ \begin{array}{l} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x \pi(\tau, V) = 0, \\ \partial_t V = \frac{1}{\epsilon}(\tau - V), \end{array} \right. \tag{3}$$

where V is a new variable, so-called relaxation variable, and the function π is defined, for a fixed number $a > 0$, by

$$\pi(\tau, V) = p(V) + a^2(V - \tau). \tag{4}$$

1. Show formally that, when the parameter ϵ tends to 0, system (3) leads to (1) in the limit.

2. Define the function

   $$S^\epsilon(\tau, u, V) = \frac{1}{2}u^2 + E(V) + \frac{\pi^2 - p^2(V)}{2a^2},$$

   where E is a primitive of $-p($, namely $E'(V) = -p(V)$. Prove that $S^\epsilon(\tau, u, V)$ is a "relaxed" entropy of system (3) in the sense that, for a smooth solution,

   $$\partial_t S^\epsilon + \partial_x(\pi u) = -\frac{1}{\epsilon}(a^2 + p'(V))(\tau - V)^2. \tag{5}$$

   Deduce that, if a is large enough, the relaxed entropy is dissipated by (3).

3. From now on, for all questions in the sequel, **one shall ignore the source term** (without derivatives) in the third equation of (3) which simply becomes $\partial_t V = 0$. Prove that system (3) is strictly hyperbolic for $\tau > 0$ and $u, V \in \mathbb{R}$.

4. Prove that the three characteristic fields are linearly degenerates (LD). Which type of waves are associated to these LD fields ?

5. Find two (independent) Riemann invariants for each characteristic field. What is the main property of these Riemann invariants for the waves associated to their LD fields ?

6. Plot in the (π, u) plane the wave curves for the 1 and 3 fields (as usual, fields are labelled by increasing order of their eigenvalues). What can be said about the wave curve for the 2 field in the (π, u) plane ? Recall that the wave curve is the set of all right states (π_R, u_R, V_R) which can be connected to a given left state (π_L, u_L, V_L).

7. What can be said about the entropy condition for the LD fields of system (3) ?

8. We consider the Riemann problem for (3) with right (π_R, u_R, V_R) and left (π_L, u_L, V_L) states. With the help of the intersection points of the wave curves in (π, u) plane, prove that the solution of the Riemann problem is made of 4 constant states, separated by 3 waves (at most) and give their precise values (states and wave speeds).

9. Deduce a numerical method for solving (3).