

Continuous (convex) optimisation

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Lecture 2: fixed point iterations; convexity.

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Let $T : \mathcal{X} \rightarrow \mathcal{X}$ (\mathcal{X} is a Banach or Hilbert space) which is “nonexpansive” or 1 Lipschitz:

$$|Tx - Ty| \leq |x - y| \quad \forall x, y \in \mathcal{X}.$$

We consider the problem of finding a fixed point $Tx = x$.

If in addition it is ρ -Lipschitz with $\rho < 1$, then Picard's classical fixed point theorem shows that the iterates $x^k = T^k x^0$, $k \geq 1$, form a Cauchy sequence and therefore converge to a fixed point, necessarily unique. (This relies on the fact that the space is complete.)

If $\rho = 1$, then the iterations may, or may not, converge. For instance, if $Tx = -x$, then $T0 = 0$ but $x^k = (-1)^k x_0$ will never converge unless $x_0 = 0$.

Definition: Averaged operator

A nonexpansive operator is *averaged* if it is of the form

$$(1 - \theta)I + \theta T$$

where $\theta \in [0, 1)$ and T is 1-Lipschitz.

Remark: for $\theta = 0$, it is I , which is averaged but not very interesting.

Example: $x \mapsto x - \tau \nabla f(x)$ where f is convex, with L -Lipschitz gradient, and $0 \leq \tau < 2/L$.

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Setting: T a nonexpansive operator in a *Hilbert* space \mathcal{X} .
 $F = \{x \in \mathcal{X} : Tx = x\} \neq \emptyset$. $T_\theta := (1 - \theta)I + \theta T$.

Theorem

Let $x_0 \in \mathcal{X}$, $0 < \theta < 1$, and for $k \geq 1$, $x^k = T_\theta^k x_0$. Then there exists $x \in F$, $x^k \rightarrow x$.

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Proof: Step 1.: We remark that $x^* \in F$ if and only if $T_\theta x^* = x^*$. In particular,

$|x^{k+1} - x^*| = |T_\theta x^k - T_\theta x^*| \leq |x^k - x^*|$. We say that the sequence (x^k) is “Fejér”-monotone with respect to the set F . Then there exists, for any $x^* \in F$, $m(x^*) = \inf_k |x^k - x^*| = \lim_k |x^k - x^*|$. If $m(x^*) = 0$ for some x^* we are done, hence we may assume $m(x^*) > 0$ for all $x^* \in F$.

Step 2: We show that $x^{k+1} - x^k \rightarrow 0$: the operator T_θ is said to be “asymptotically regular”. It follows from the “parallelogram identity”. One has

$$x^{k+1} - x^* = (1 - \theta)(x^k - x^*) + \theta(T_1 x^k - x^*).$$

Hence for all k :

$$\begin{aligned} |x^{k+1} - x^*|^2 &= (1 - \theta)|x^k - x^*|^2 + \theta|T_1 x^k - x^*|^2 - \theta(1 - \theta)|T_1 x^k - x^k|^2 \\ &\leq |x^k - x^*|^2 - \frac{1 - \theta}{\theta}|x^{k+1} - x^k|^2 \end{aligned}$$

from which one deduces that $\sum_k |x^{k+1} - x^k|^2 < \infty$, hence the result.

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Remark: rate. In addition, one observes that the sequence $\frac{1-\theta}{\theta}|x^{k+1} - x^k|^2$ (which is nonincreasing) is controlled in the following way:

$$\frac{1-\theta}{\theta}(k+1)|x^{k+1} - x^k|^2 \leq \frac{1-\theta}{\theta} \sum_{i=0}^k |x^{i+1} - x^i|^2 \leq |x^0 - x^*|^2 - |x^{k+1} - x^*|^2.$$

As $x^{k+1} - x^k = \theta(T_1 x^k - x^k)$ we obtain a rate for the error $T_1 x^k - x^k$, in the Hilbertian setting, given by:

$$|T_1 x^k - x^k| \leq \frac{|x^0 - x^*|}{\sqrt{\theta(1-\theta)}\sqrt{k+1}}.$$

In fact, Cominetti-Soto-Vaisman showed recently (2014) that in any normed space

$$|x^k - T x^k| \leq \frac{1}{\sqrt{\pi}} \frac{|x^0 - x^*|}{\sqrt{\sum_{i=1}^k \theta_i(1-\theta_i)}}$$

(if θ is allowed to vary at each step, and assuming there exists a fixed point x^* , their result is a bit more general).

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Step 3 relies on:

Opial's lemma

If in a Hilbert space \mathcal{X} the sequence $(x_n)_n$ is weakly convergent to x_0 then for any $x \neq x_0$,

$$\liminf_n |x_n - x| > \liminf_n |x_n - x_0|$$

Proof:

$$|x_n - x|^2 = |x_n - x_0|^2 + 2 \langle x_n - x_0, x_0 - x \rangle + |x_0 - x|^2.$$

Since $\langle x_n - x_0, x_0 - x \rangle \rightarrow 0$ by weak convergence, we deduce

$$\liminf_n |x_n - x|^2 = \liminf_n (|x_n - x_0|^2 + |x_0 - x|^2) = |x_0 - x|^2 + \liminf_n |x_n - x_0|^2$$

and the claim follows. □

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Step 3: since T_θ is nonexpansive, for each k ,

$$\begin{aligned} |x^k - \bar{x}| &\geq |T_\theta x^k - T_\theta \bar{x}| \\ &= |x^{k+1} - x^k + x^k - T_\theta \bar{x}| \geq |x^k - T_\theta \bar{x}| - |x^{k+1} - x^k|. \end{aligned}$$

Let x^{k_l} be a weakly converging subsequence of x^k (which is bounded by step 1). We deduce (thanks to the previous Step 2):

$$\liminf_l |x^{k_l} - \bar{x}| \geq \liminf_l |x^{k_l} - T_\theta \bar{x}|.$$

Opial's lemma implies that $T_\theta \bar{x} = \bar{x}$.

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Step 4: To conclude, assume that a subsequence $(x^{m_l})_l$ of $(x^k)_k$ converges weakly to another fixed point $\bar{y} \neq \bar{x}$.

Then by Opial's lemma we deduce both that $m(\bar{x}) < m(\bar{y})$ and $m(\bar{y}) < m(\bar{x})$:

$$m(\bar{y}) = \liminf_l |x^{m_l} - \bar{y}| < \liminf_l |x^{m_l} - \bar{x}| = m(\bar{x}),$$

$$m(\bar{x}) = \liminf_l |x^{k_l} - \bar{x}| < \liminf_l |x^{k_l} - \bar{y}| = m(\bar{y}),$$

a contradiction. It follows that the whole sequence (x^k) must weakly converge to \bar{x} . □

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Varying steps: One can consider more generally iterations of the form

$$x^{k+1} = x^k + \tau_k(T_1 x^k - x^k)$$

with varying steps τ_k . Then, if $0 < \underline{\tau} \leq \tau_k \leq \bar{\tau} < 1$, the convergence still holds, with almost the same proof.

Remark: A sufficient condition is that $\sum_k \tau_k(1 - \tau_k) = \infty$, see (Reich, 1979).

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Assume now the sequence (x_k) is an “inexact” iteration of T_θ :

$$|x^{k+1} - T_\theta x^k| \leq \varepsilon_k.$$

Then one has the following result:

Theorem

If $\sum_k \varepsilon_k < \infty$, then $x^k \rightarrow \bar{x}$ a fixed point of T (if one exists).

(The condition is quite strong. [but clearly necessary!])

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Proof: now, x^k is “quasi-Fejér monotone”: denoting $e_k = x^{k+1} - T_\theta x^k$ so that $|e_k| \leq \varepsilon_k$,

$$|x^{k+1} - x^*| = |T_\theta x^k - T_\theta x^* + e_k| \leq |x^k - x^*| + \varepsilon_k$$

for all k , and any $x^* \in F$. Hence, $|x^{k+1} - x^*| \leq |x^0 - x^*| + \sum_{i=0}^k \varepsilon_i$ is bounded. Letting $a_k = \sum_{i=k}^{\infty} \varepsilon_i$ which is finite and goes to 0 as $k \rightarrow \infty$, this can be rewritten

$$|x^{k+1} - x^*| + a_{k+1} \leq |x^k - x^*| + a_k$$

so that once more one can define

$$m(x^*) := \lim_{k \rightarrow \infty} |x^k - x^*| = \inf_{k \geq 0} |x^k - x^*| + a_k$$

Again, if $m(x^*) = 0$ the theorem is proved. If not, the rest of the proof follows steps 2, 3, 4 with very little changes.

Krasnoselskii-Mann's convergence theorem

Gradient Descent for Convex functions

It follows from this Theorem the convergence for the explicit and implicit gradient descent for convex functions.

Consider $T_\tau(x) = x - \tau \nabla f(x)$ for f convex with L -Lipschitz gradient. We have that

$$T_{2/L}(x) = x - \frac{2}{L} \nabla f(x)$$

is nonexpansive. Hence if $0 < \tau < 2/L$, one has

$$T_\tau(x) = x - \frac{\tau L}{2} \frac{2}{L} \nabla f(x) = \frac{\tau L}{2} T_{2/L}(x) + \left(1 - \frac{\tau L}{2}\right) x$$

is an averaged operator with $\theta = L\tau/2 \in]0, 1[$.

Hence if $x^k = T_\tau^k x_0$, and T_τ has a fixed point, x^k weakly converges to a fixed point. Now, $T_\tau x = x$ if and only if $\nabla f(x) = 0$. We have $|\nabla f(x^k)| \lesssim 1/\sqrt{k}$.

(For the implicit descent, if f is convex, we will see that the associated operator is always $1/2$ -averaged.)

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Composition of Averaged Operators

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Important remark: Let T_θ, S_λ be averaged operators: $T_\theta = (1 - \theta)I + \theta T_1$, $S_\lambda = (1 - \lambda)I + \lambda S_1$. Then $T_\theta \circ S_\lambda$ is also averaged: letting $\mu = \theta + \lambda(1 - \theta) \in]0, 1[$, one has

$$T_\theta \circ S_\lambda = (1 - \mu)I + \mu \frac{(1 - \theta)\lambda S_1 + \theta T_1 \circ ((1 - \lambda)I + \lambda S_1)}{\theta + (1 - \theta)\lambda}.$$

Application: “Forward-Backward splitting”. Show that if, for g convex and f convex with L -Lipschitz gradient

$$\min_x f(x) + g(x)$$

has a solution x^* then

$$x^{k+1} = \arg \min_x \frac{|x - (x - \tau \nabla f(x))|^2}{2\tau} + g(x)$$

weakly converges to a solution if $0 < \tau < 2/L$.

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A **convex** set C in a linear space is such that $x, y \in C \Rightarrow [x, y] \in C$ where $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$.

A **convex function** is a function $f : C \rightarrow \mathbb{R}$ such that for any $x, y \in C$, $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

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A “*generalized*” *convex function* is a function $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ such that its *epigraph*

$$\text{epi } f := \{(x, \lambda) \in \mathcal{X} \times \mathbb{R} : \lambda \geq f(x)\}$$

is a convex set.

We define the *domain* of f as $\text{dom } f = \{x : f(x) < +\infty\}$. It is (clearly) a convex set.

Then, f is *proper* if it is not identically $+\infty$ and never $-\infty$. In this case, again, one can say it is convex if and only if for any $x, y \in \mathcal{X}$, $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

where now some values here may be $+\infty$.

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Remark: If there exists \check{x} with $f(\check{x}) = -\infty$, then for any $x \in \text{dom } f$, as soon as there exists $z \in \text{dom } f$ with $x \in [\check{x}, z]$, one has for some $t \in (0, 1)$:

$$f(x) \leq tf(\check{x}) + (1 - t)f(z) = -\infty.$$

In particular, f is identically $-\infty$ in the relative interior $\text{ri dom } f$ of its domain, where for a convex C , the *relative interior of C* is defined as:

$$\text{ri } C := \{x \in C : \forall y \in C \setminus \{x\}, \exists z \in C \setminus \{x\}, x \in [y, z]\}.$$

(In finite dimension, this is nothing but the interior of C in the subspace spanned by C .)

Hence, convex functions which take the value $-\infty$ are not particularly interesting...

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f is *strictly convex* if the inequality is strict whenever $x \neq y$ and $0 < t < 1$.

If \mathcal{X} is a normed space, f is *strongly convex* (or μ -convex) if in addition, there exists $\mu > 0$ such that for all $x, y \in \mathcal{X}$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mu \frac{t(1-t)}{2} |x - y|^2.$$

[This can be used also with non-Hilbertian norms!]

Thanks to the parallelogram identity, *in the Hilbertian setting*, one easily checks that this is equivalent to require that $x \mapsto f(x) - \mu/2|x|^2$ is still convex. The archetypical example of a μ -convex function is a quadratic plus affine function $\mu|x|^2/2 + \langle b, x \rangle + c$.

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The function f is *lower semi-continuous* (l.s.c.) if for all $x \in \mathcal{X}$, if $x_n \rightarrow x$, then

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

It is easy to see that f is l.s.c. if and only if $\text{epi } f$ is closed.

Example: An important example of a convex function the *characteristic function* or *indicator function* of a set C (often denoted ι_C , χ_C , or δ_C):

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{else,} \end{cases}$$

which is convex and proper as soon as C is convex and non-empty, and is l.s.c. as soon as C is closed.

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Lemma

Let f be proper, convex.

If there exists $B \subset \text{dom } f$ a ball with $\sup_B f < +\infty$, then f is locally Lipschitz in the interior of $\text{dom } f$.

In finite dimension, a proper convex function f is locally Lipschitz in $\text{ri dom } f$.

Proof: Wlog assume $B = B(0, \delta)$, $\delta > 0$, let $M = \sup_B f < \infty$.

For $x \in B$, by convexity $f(x) \geq 2f(0) - f(-x) \geq 2f(0) - M$, hence $|f| \leq M + 2|f(0)|$.

We prove that f is Lipschitz in $B(0, \delta/2)$: given $x, y \in B(0, \delta/2)$, there is $z \in B(0, \delta)$ such that $y = (1-t)x + tz$ for some $t \in [0, 1]$, and $|z-x| \geq \delta/2$.

By convexity, $f(y) - f(x) \leq t(f(z) - f(x)) \leq t2(M - f(0))$. We have $t(z-x) = y-x$ so that $t \leq |y-x|/|z-x| \leq 2|y-x|/\delta$, so that

$$f(y) - f(x) \leq \left(\frac{4(M - f(0))}{\delta} \right) |y - x|.$$

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Now let \bar{x} in the interior of $\text{dom } f$. Observe that for some $\lambda > 1$, $\lambda\bar{x} \in \text{dom } f$ hence

$B' = 1/\lambda(\lambda\bar{x}) + (1 - 1/\lambda)B(0, \delta) = B(\bar{x}, \delta(1 - 1/\lambda)) \subset \text{dom } f$; moreover, if

$x \in B'$, $x = 1/\lambda(\lambda\bar{x}) + (1 - 1/\lambda)z$ for some z with $f(z) \leq M$ hence $f(x) \leq 1/\lambda f(\lambda\bar{x}) + (1 - 1/\lambda)M$, so

that $\sup_{B'} f < \infty$. Hence as before f is Lipschitz in a smaller ball. This shows that f is locally Lipschitz in the interior of $\text{dom } f$.

In finite dimension, assume $0 \in \text{dom } f$ and let d be the dimension of $\text{vect dom } f$. It means there exist x_1, \dots, x_d independent points in $\text{dom } f$. Now, the d -dimensional set $\{\sum_i t_i x_i : t_i > 0, \sum_i t_i \leq 1\}$ (the interior of the convex envelope of $\{0, x_1, \dots, x_d\}$) is an open set in $\text{vect dom } f$, moreover if $x = \sum_i t_i x_i$, $f(x) \leq \sum_i t_i f(x_i) + (1 - \sum_i t_i) f(0) \leq M := \max\{f(0), f(x_1), \dots, f(x_d)\}$. Hence we can apply the first part of the theorem, and f is locally Lipschitz in the relative interior of the domain.

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In infinite dimension one can possibly find noncontinuous linear forms hence noncontinuous convex functions. (the typical example is a linear function defined by $f(e_n) = n$ where $(e_n)_{n \geq 1}$ is an independent family, which is then completed into a basis \mathcal{B} , then, one lets $f(e) = 0$ if $e \in \mathcal{B} \setminus \{e_n : n \geq 1\}$.) [Hence: such a function cannot be bounded on an open set!]

A convex proper *lower semi-continuous* function is always locally bounded in the interior of its domain, and therefore locally Lipschitz.

Indeed if 0 is an interior point and one considers the convex closed set $C = \{x : f(x) \leq 1 + f(0)\}$, one can check that $\bigcup_{n \geq 1} nC = \mathcal{X}$, as if $x \in \mathcal{X}$, $t \mapsto f(tx)$ is locally Lipschitz near $t = 0$. Hence $\overset{\circ}{C} \neq \emptyset$ by Baire's property: it follows that there is an open ball where f is bounded, as requested (see Ekeland-Temam, Cor. 2.5).

But in infinite dimension there are also many interesting lsc. convex functions whose domain has empty interior.

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Convex sets are easily “separated” by hyperplanes. This is called a geometric version of Hahn-Banach’s theorem. In the Hilbert settings, the proofs are quite easy, and constructive (why in more general settings, the proofs usually rely on Zorn’s lemma).

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Convex sets are easily “separated” by hyperplanes. This is called a geometric version of Hahn-Banach’s theorem. In the Hilbert settings, the proofs are quite easy, and constructive (why in more general settings, the proofs usually rely on Zorn’s lemma).

Separation Theorem (1st version)

Let \mathcal{X} be a (real) Hilbert space, $C \subset \mathcal{X}$ a **closed**, convex set and $x \notin C$. Then there exists a closed hyperplane which “separates” strictly x and C :

$\exists v \in \mathcal{X}, \alpha \in \mathbb{R}$ s.t.:

$$\langle v, x \rangle > \alpha \geq \langle v, y \rangle \quad \forall y \in C$$

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Proof: We introduce the projection $z = \Pi_C(x)$ defined by $|x - z| = \min_{y \in C} |x - y|$.
The first order optimality condition for z is found by writing that for any $y \in C$,

$$|x - z|^2 \leq |x - (z + t(y - z))|^2 \quad \forall t \in (0, 1].$$

Sending $t \rightarrow 0$, we find:

$$\langle x - z, y - z \rangle \leq 0 \quad \forall y \in C.$$

Hence if $v = x - z \neq 0$, $y \in C$,

$$\langle v, x \rangle = \langle x - z, x \rangle = |x - z|^2 + \langle x - z, z \rangle \geq |x - z|^2 + \langle x - z, y \rangle = |v|^2 + \langle v, y \rangle.$$

The result follows (letting for instance $\alpha = |v|^2/2 + \sup_{y \in C} \langle v, y \rangle$).

The proof can easily be extended to the situation where $\{x\}$ is replaced with a compact convex set K not intersecting C : consider $C' = C - K$ and $0 \notin C'$

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Corollary

In a real Hilbert space \mathcal{X} , a closed convex set C is weakly closed.

Indeed, if $x \notin C$, one finds v, α with $\langle v, x \rangle > \alpha \geq \langle v, y \rangle$ for all $y \in C$ and this defines a neighborhood $\{\langle v, \cdot \rangle > \alpha\}$ of x for the weak topology which does not intersect C : the complement of C is therefore open.

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Separation Theorem (2nd version)

Let \mathcal{X} be a real Hilbert space, $C \subset \mathcal{X}$ an **open** convex set and $C' \subset \mathcal{X}$ a convex set with $C' \cap C = \emptyset$. Then there exists a closed hyperplane which "separates" C and C' :

$\exists v \in \mathcal{X}, \alpha \in \mathbb{R}, v \neq 0, \text{ s.t.}:$

$$\langle v, x \rangle \geq \alpha \geq \langle v, y \rangle \quad \forall x \in C, y \in C'$$

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Proof: first case: $C' = \{\bar{x}\}$ is a singleton.

The difficult case is whenever $\bar{x} \in \bar{C} \setminus C$, otherwise we can apply the previous theorem to separate (strictly) \bar{x} and \bar{C} .

By assumption, there exists a ball $B = B(y, \delta) \subset C$.

Let $x_n = y + (1 + \frac{1}{n})(\bar{x} - y)$, which is such that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Since

$$\bar{x} = \frac{n}{n+1}x_n + \frac{1}{n+1}y,$$

one has $x_n \notin \bar{C}$. Indeed, if $x_n \in \bar{C}$, there is $x' \in C$ with $|x' - x_n| < \frac{\delta}{2n}$, and by convexity, $C \supset \frac{n}{n+1}x' + \frac{1}{n+1}B(y, \delta) = B(\frac{n}{n+1}(x' - x_n) + \bar{x}, \frac{\delta}{n+1}) \ni \bar{x}$, a contradiction.

By the previous separation Theorem, there exists v_n such that for all $x \in \bar{C}$,

$$\langle v_n, x_n \rangle \leq \langle v_n, x \rangle$$

and we can assume $|v_n| = 1$. Up to a subsequence, we may even assume that $v_n \rightarrow v$ weakly in \mathcal{X} .

In the limit, (using that $x_n \rightarrow \bar{x}$ strongly) we obtain $\langle v, \bar{x} \rangle \leq \langle v, x \rangle \forall x \in C$, which is our claim if $v \neq 0$ (for instance, in finite dimension).

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Using again the ball $B(y, \delta) \subset C$, one has for any $|z| \leq 1$

$$\langle v_n, x_n \rangle \leq \langle v_n, y - \delta z \rangle$$

so that $\langle v_n, y - x_n \rangle \geq \delta \langle v_n, z \rangle$.

We consider the sup over all possible z : we find $\langle v_n, y - x_n \rangle \geq \delta$. In the limit we deduce $\langle v, y - \bar{x} \rangle \geq \delta$ which shows that $v \neq 0$.

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We consider the sup over all possible z : we find $\langle v_n, y - x_n \rangle \geq \delta$. In the limit we deduce $\langle v, y - \bar{x} \rangle \geq \delta$ which shows that $v \neq 0$.

Now, to show the general case, one lets $A = C' - C = \{y - x : y \in C', x \in C\}$: this is an open convex set and by assumption, $0 \notin A$. Hence by the previous part, there exists $v \neq 0$ such that $\langle v, y - x \rangle \leq \langle v, 0 \rangle = 0$ for all $y \in C', x \in C$, which is what we wanted to show. □

Convex duality: Legendre-Fenchel conjugate

(Or “convex conjugate”)

Consider a function $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$.

Definition: Legendre-Fenchel conjugate

We define, for any $y \in \mathcal{X}$,

$$f^*(y) := \sup_{x \in \mathcal{X}} \langle y, x \rangle - f(x)$$

We also define the *bi-conjugate* of f^* as $f^{**} = (f^*)^*$.

Remark: in a general vector space, f^* is defined for $y \in \mathcal{X}^*$ (a dual space).

Remark: f^* is defined as a sup of continuous, linear forms. It is therefore lower-semicontinuous (as a sup of continuous functions — including for the weak topology) and convex (as a sup of convex functions).

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Proposition

Let $f : \mathcal{X} \rightarrow [-\infty, +\infty]$. Then $f^{**} \leq f$.

This is because, by definition, for any x, y ,

$$f^*(y) \geq \langle y, x \rangle - f(x) \Leftrightarrow f(x) + f^*(y) \geq \langle y, x \rangle \Leftrightarrow f(x) \geq \langle y, x \rangle - f^*(y).$$

Taking the supremum with respect to y , we find that $f \geq f^{**}$. □

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Remark: Given $y \in \mathcal{X}$, if for any $c \in \mathbb{R}$, there is $x \in \mathcal{X}$ with $f(x) \leq \langle y, x \rangle - c$, then $f^*(y) = +\infty$. On the other hand, if there exists c such that $f(x) \geq \langle y, x \rangle - c$ for all x , then $f^*(y) \leq c < +\infty$.

So $f^* \not\equiv +\infty$ if and only if f is larger than at least an affine function. If not, $f^* \equiv +\infty$ and $f^{**} \equiv -\infty$.

If f is larger than at least one affine function, and $f \not\equiv +\infty$, then $f^* \not\equiv +\infty$ and $f^*(y) > +\infty$ for all y : f^* is proper. In this case, also f^{**} is proper and one has:

Theorem - convex bi-conjugate

*Let $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$, and assume there is an affine function below f . Then f^{**} is the largest convex, lsc. function below f .*

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The result will be a consequence of the following:

Theorem

*Let $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex, proper, lower-semicontinuous. Then $f^{**} = f$.*

(Actually, both results are corollaries of the other.)

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The result will be a consequence of the following:

Theorem

Let $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex, proper, lower-semicontinuous. Then $f^{**} = f$.

(Actually, both results are corollaries of the other.)

Proof: If not, there is $(x, t) \in \text{epi } f^{**}$ with $f^{**}(x) \leq t < f(x)$, that is $(x, t) \notin \text{epi } f$. Since $\text{epi } f$ is convex, closed, thanks to the first separation theorem, $\exists (v, \lambda) \in \mathcal{X} \times \mathbb{R}, \alpha \in \mathbb{R}$, s.t.:

$$\langle v, x \rangle - \lambda t > \alpha \geq \langle v, x' \rangle - \lambda t' \quad \forall (x', t') \in \text{epi } f.$$

Since one can send $t' \rightarrow -\infty$, we see that λ must be non-negative. If $\lambda \neq 0$, one has, letting $y = v/\lambda$,

$$\langle y, x \rangle - t > \frac{\alpha}{\lambda} \geq \langle y, x' \rangle - f(x')$$

for any $x' \in \text{dom } f$, so that $\alpha/\lambda \geq f^*(y)$. We deduce $f^{**}(x) \leq t < \langle y, x \rangle - f^*(y)$, which is a contradiction.

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It remains the case $\lambda = 0$. One has $\langle v, x \rangle > \alpha \geq \langle v, x' \rangle \quad \forall x' \in \text{dom } f$.

We remark that f being convex, lsc, proper, is above an affine function: indeed given a point $\bar{x} \in \text{dom } f$, the separation theorem applied to $(\bar{x}, f(\bar{x}) - 1) \notin \text{epi } f$ shows that there exists (w, μ) such that

$$\langle w, \bar{x} \rangle - \mu(f(\bar{x}) - 1) > \langle w, x' \rangle - \mu f(x')$$

for all $x' \in \text{dom } f$. Taking $x' = \bar{x}$ we see that $\mu > 0$, hence letting $p = w/\mu$, we obtain $f(x') > \langle p, x' \rangle - c$ for $c = f(\bar{x}) - 1 - \langle p, \bar{x} \rangle$.

Hence for $t > 0$, one has:

$$f^*(tv + p) = \sup_{x'} t \langle v, x' \rangle + \langle p, x' \rangle - f(x') \leq t \sup_{x' \in \text{dom } f} \langle v, x' \rangle + c \leq \alpha t + c,$$

so that

$$f^{**}(x) = \sup_q \langle q, x \rangle - f^*(q) \geq \sup_{t>0} \langle tv + p, x \rangle - f^*(tv + p) \geq \sup_{t>0} t(\langle v, x \rangle - \alpha) + \langle p, x \rangle - c = +\infty$$

which is again a contradiction since we assumed $f^{**}(x) < f(x)$. □

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Now we prove the first theorem, that is the case where f is not necessarily convex, lower-semicontinuous. In this case, we use that as before $f^{**} \leq f$. If $f \equiv +\infty$ then $f^* \equiv -\infty$ and $f^{**} \equiv +\infty$: the result is trivial. Otherwise $f^{**} \leq f$ is proper.

Consider $g \leq f$ convex, lower-semicontinuous. We want to show that $g \leq f^{**}$. We may assume g is proper, otherwise $g \leq g' = \max\{g, a\} \leq f$, where a is some affine function below f , g' is proper, and $g' \leq f^{**} \Rightarrow g \leq f^{**}$.

One has that $g^* \geq f^*$ and $g^{**} \leq f^{**}$. By the last theorem, $g^{**} = g$, so that $g \leq f^{**}$. Hence f^{**} is the largest convex, semi-continuous function below f . □

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Remark: the following “local” version of the last result can also be useful.

Corollary

*Let f be convex, proper and assume it is lower-semicontinuous at x . Then $f^{**}(x) = f(x)$.*

Proof: $f(x) \leq \liminf_{y \rightarrow x} f(y)$, equivalently for any $t < f(x)$, there is a ball $B = B(x, \delta)$ with $f > t$ in B . Hence $\text{epi } f \cap (B(x, \delta) \times (-\infty, t)) = \emptyset$, so that also $\overline{\text{epi } f} \cap (B(x, \delta) \times (-\infty, t)) = \emptyset$. Being f convex, $\overline{\text{epi } f} = \text{epi } f^{**}$ hence $f^{**}(x) \geq t$. Letting $t \rightarrow f(x)$ we deduce $f^{**}(x) \geq f(x)$ □

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- ① $f(x) = |x|^2/(2\alpha)$, $\alpha > 0$: $f^*(y) = \alpha|y|^2/2$;
- ② $f(x) = |x|^p/p$: $f^*(y) = |y|^{p'}/p'$, $1/p + 1/p' = 1$;
- ③ $F(f) = \|f\|_{L^p}^p/p$: $F^*(g) = \|g\|_{L^{p'}}^{p'}/p'$ (the duality is in L^2 , however this is also true in the $(L^p, L^{p'})$ duality, cf Ekeland-Temam's book);
- ④ $f(x) = \delta_{B(0,1)}(x) = 0$ if $x \in B(0, 1)$, $+\infty$ else: $f^*(p) = |p|$.

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If f is positively 1-homogeneous, then

$$f^*(y) = \sup_x \langle y, x \rangle - f(x) = \sup_{t>0} \sup_x \langle y, tx \rangle - f(tx) = \sup_{t>0} tf^*(y) \in \{0, +\infty\}$$

and precisely

$$f^*(y) = \begin{cases} 0 & \text{if } \langle y, x \rangle \leq f(x) \forall x \in \mathcal{X}, \\ +\infty & \text{if } \exists x \in \mathcal{X}, \langle y, x \rangle > f(x). \end{cases}$$

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and precisely

$$f^*(y) = \begin{cases} 0 & \text{if } \langle y, x \rangle \leq f(x) \forall x \in \mathcal{X}, \\ +\infty & \text{if } \exists x \in \mathcal{X}, \langle y, x \rangle > f(x). \end{cases}$$

Letting $C = \{y : \langle y, x \rangle \leq f(x) \forall x \in \mathcal{X}\}$ one has $f^* = \delta_C$ (C is clearly closed and convex, and f^* convex lsc). Eventually, observe that if f is convex and lsc, then $f^{**} = f$ which shows that in this case:

$$f(x) = \sup_{y \in C} \langle y, x \rangle.$$

Conjugate: “ ∞ -homogeneous” functions / β -homogeneous functions.

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Conversely if $f = \delta_C$ for some set C (f is the *support function* of C), one easily sees that $f^*(y) = \sup_{x \in C} \langle y, x \rangle$ is convex, 1-homogeneous, and that $f^{**} = \delta_{\text{co}C}$ the characteristic of the closed convex envelope of C , that is the smallest closed convex set containing C .

Conjugate: “ ∞ -homogeneous” functions / β -homogeneous functions.

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In general for f positively β -homogeneous, $1 < \beta < \infty$, then

$$f^*(ty) = \sup_x \langle ty, x \rangle - f(x) = t^\alpha \sup_x \langle y, t^{1-\alpha}x \rangle - f(t^{-\alpha/\beta}x) = t^\alpha f^*(y)$$

if $1 - \alpha = -\alpha/\beta$, hence if $1/\alpha + 1/\beta = 1$.

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We consider f, g convex lsc. functions. We define their *inf-convolution* as

$$f \square g(x) = \inf_y f(x - y) + g(y).$$

This defines a convex function (more generally, given $G(x, y)$ convex in (x, y) , $x \mapsto \inf_y G(x, y)$ is also convex).

Lemma

If there is $p \in \mathcal{X}$ where f^ is continuous and g^* is finite, then the inf is a “min” and $f \square g$ is convex, lsc. In finite dimension, it is enough to have $p \in \text{ri dom } f^* \cap \text{ri dom } g^*$.*

Inf-convolutions: proof

Proof: consider indeed $x_n \rightarrow x$ and y_n such that

$$f \square g(x_n) \geq f(x_n - y_n) + g(y_n) - \frac{1}{n}.$$

Consider a subsequence with

$$\lim_k f(x_{n_k} - y_{n_k}) + g(y_{n_k}) = \liminf_n f(x_n - y_n) + g(y_n) \leq \liminf_n f \square g(x_n)$$

Observe that if f^* is continuous at p , then it means that there is a constant c such that

$$f^*(q) \leq c + \delta_{B(0,\varepsilon)}(q - p)$$

(where δ_C is the characteristic function of C which is zero in C and $+\infty$ elsewhere) while $g^*(p) < +\infty$: so that for all z

$$f(z) = f^{**}(z) \geq \langle p, z \rangle - c + \varepsilon|z|, \quad g(z) \geq \langle p, z \rangle - g^*(p).$$

Hence,

$$\begin{aligned} f(x_{n_k} - y_{n_k}) + g(y_{n_k}) &\geq \langle p, x_{n_k} - y_{n_k} \rangle - c + \varepsilon|x_{n_k} - y_{n_k}| + \langle p, y_{n_k} \rangle - g^*(p) \\ &= \langle p, x_{n_k} \rangle + \varepsilon|x_{n_k} - y_{n_k}| - (c + g^*(p)) \end{aligned}$$

so that $(x_{n_k} - y_{n_k})_k$ is a bounded sequence, hence there exists y and a subsequence of (y_{n_k}) (not relabelled) with $y_{n_k} \rightarrow y$. In the limit (as, f, g are weakly lsc),

$$f \square g(x) \leq f(x - y) + g(y) \leq \liminf_k f(x_{n_k} - y_{n_k}) + g(y_{n_k}) \leq \liminf_n f \square g(x_n).$$

If the sequence $x_n \equiv x$, then this proves that there is a minimizer y in the definition of the inf-convolution.

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Corollary

Let f, g be convex, lsc: if there exists $x \in \text{dom } f \cap \text{dom } g$ such that f is continuous at x (in finite dimension, $x \in \text{ri dom } f \cap \text{ri dom } g$), then $(f + g)^* = f^* \square g^*$,

Proof: By our assumption and the previous result, $f^* \square g^*$ is lsc, and:

$$\begin{aligned}(f^* \square g^*)^*(x) &= \sup_{p, q} \langle x, p \rangle - f^*(q) - g^*(p - q) \\ &= \sup_{p, q} \langle x, q \rangle - f^*(q) + \langle x, p - q \rangle - g^*(p - q) = f(x) + g(x).\end{aligned}$$

Hence $(f + g)^* = (f^* \square g^*)^{**} = f^* \square g^*$.

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Now we consider the problem

$$\min_{x \in \mathcal{X}} f(Kx) + g(x)$$

with $K : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous linear map and f, g convex, lsc. Then:

$$\begin{aligned} (\mathcal{P}) &= \min_x f(Kx) + g(x) = \min_x \sup_y \langle y, Kx \rangle - f^*(y) + g(x) \\ &\geq \sup_y \inf_x \langle K^*y, x \rangle + g(x) - f^*(y) = \sup_y -(g^*(-K^*y) + f^*(y)) = (\mathcal{D}) \end{aligned}$$

If there is equality one says that there is “Strong duality”. This is “often” true. The problem “ (\mathcal{P}) ” is usually called the *primal problem* and “ (\mathcal{D}) ” the *dual problem*.

Krasnoselskii-
Mann's
convergence
theorem

Averaged operators

Krasnoselskii-Mann's
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Extensions

Examples

Convex
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monotone
operators.

Convex analysis

Separation and
duality

Fenchel-Rockafellar
duality

Fenchel-Rockafellar duality

Continuous
(convex)
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The *primal-dual gap*

$$\mathcal{G}(x, y) = f(Kx) + g(x) + g^*(-K^*y) + f^*(y)$$

is a measure of optimality. It vanishes at (x^*, y^*) , if and only if $(\mathcal{P}) = (\mathcal{D})$, x^* is optimal in (\mathcal{P}) and y^* in (\mathcal{D}) , (x^*, y^*) is a *saddle point* of the *Lagrangian*

$$\mathcal{L}(x, y) = \langle y, Kx \rangle - f^*(y) + g(x),$$

as one has

$$\mathcal{L}(x^*, y) \leq \mathcal{L}(x^*, y^*) \leq \mathcal{L}(x, y^*) \tag{S}$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$. [Indeed, for all y, x ,
 $\mathcal{L}(x^*, y) \leq f(Kx^*) + g(x^*) = -f^*(y^*) - g^*(-K^*y^*) \leq \mathcal{L}(x, y^*)$.]

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The following is an example of Strong duality theorem

Theorem

If there exists $\bar{x} \in \text{dom } g$ with f continuous at $K\bar{x}$, then $(\mathcal{P}) = (\mathcal{D})$. (Moreover under these assumptions, (\mathcal{D}) has a solution.)

In finite dimension, it is shown in Rockafellar, (Cor 31.2.1) that Strong duality if there exists $x \in \text{ri dom } g$ with $Kx \in \text{ri dom } f$, or even more generally that $0 \in \text{ri}(\text{dom } f - K\text{dom } g)$ (the proof is almost the same as below).

Fenchel-Rockafellar duality

Proof: We use the so-called “perturbation method”: We introduce, for $z \in \mathcal{Y}$,

$$\Phi(z) := \inf_{x \in \mathcal{X}} f(Kx + z) + g(x).$$

Assume $\Phi(0) > -\infty$ (otherwise there is nothing to prove), then by assumption, one can find M and ε such that for $|z| < \varepsilon$, $\Phi(z) \leq f(K\bar{x} + z) + g(\bar{x}) \leq M < +\infty$. Being Φ convex, we deduce that it is locally Lipschitz near 0 and in particular thanks to a previous theorem, $\Phi(0) = \Phi^{**}(0) = \sup_y -\Phi^*(y)$. We compute:

$$\begin{aligned} \Phi^*(y) &= \sup_{z \in \mathcal{Y}} \langle y, z \rangle - \inf_{x \in \mathcal{X}} (f(Kx + z) + g(x)) \\ &= \sup_{x, z} \langle y, z + Kx \rangle - \langle K^*y, x \rangle - f(Kx + z) - g(x) = f^*(y) + g^*(-K^*y). \end{aligned}$$

The claim follows. The subdifferentiability theory (next lecture) will also show that under these assumptions, there always is a solution to the dual problem. (One can also follow the steps of the proof of minimality of inf-convolutions, for a minimizing sequence of the dual problem.)

Remark: In finite dimension, if $0 \in \text{ri}(\text{dom } f - K\text{dom } g)$, one can show again that Φ is lsc at 0 and proceed in the same way.

Fenchel-Rockafellar duality

Example

We consider the problem

$$\min_x \lambda \|Dx\|_1 + \frac{1}{2} \|x - x^0\|^2 \quad (\mathcal{P})$$

where $D : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous operator, $x^0 \in \mathbb{R}^n$, $\|\cdot\|_1$ is the ℓ^1 -norm. One has

$$f = \lambda \|\cdot\|_1, \quad K = D, \quad g = \frac{1}{2} \|\cdot - x^0\|^2.$$

Then the Lagrangian is

$$\mathcal{L}(x, y) = \langle y, Dx \rangle - f^*(y) + g(x)$$

where $f^*(y) = 0$ if $|y_i| \leq \lambda$ for $i = 1, \dots, n$, and $+\infty$ else. To find the dual problem, we compute $g^*(z) = \langle z, x^0 \rangle + \|z\|^2/2$, and we obtain

$$\max \left\{ \langle D^* y, x^0 \rangle - \frac{1}{2} \|D^* y\|^2 : |y_i| \leq \lambda, i = 1, \dots, n \right\}. \quad (\mathcal{D})$$

This can be rewritten as a projection problem:

$$\min_{|y_i| \leq \lambda} \|D^* y - x^0\|^2.$$

and can be solved for instance by *implicit* gradient descent with the metric $I/\tau - DD^*$, for $\tau < 1/\|D\|^2$.